PLANE AND SOLID GEOMETRY

WELLS

WELLS' MATHEMATICAL SERIES.

THE

ELEMENTS OF GEOMETRY.

REVISED EDITION.

BY

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PREFACE.

In the revision of the author's work on Plane and Solid Geometry, many important improvements have been effected.

With a class just commencing the study of Geometry, too much emphasis cannot be laid on the *form in which an oral or written demonstration should be presented.*

The beginner requires a certain amount of practice before he can acquire the art of putting a proof in a clear and logical form.

To give this drill, the author has, through the whole of Book I., placed directly after each step in the proof the full statement of the reason, in smaller type, enclosed in brackets.

But too much assistance of this nature is open to serious objections, as it has a tendency to make the pupil a mere automaton, and confirm him in indolent habits of study. It has seemed advisable, therefore, in Books II. to V., inclusive, to give only the number of the section where the required authority is to be found.

The above plan has been submitted to a large number of representative teachers, and in nearly every case has met with the most unqualified approval.

In the Solid Geometry, references are given in full in the first sixteen propositions of Book VI., and by section numbers only through the remainder of the work. On pages ix, x, and xi of the Introduction will be found a few propositions put in a form which is recommended for blackboard work.

Particular attention has been given to the arrangement of the propositions and corollaries in a form for convenient reference. The statement of the corollary has in every case been printed in italics; and in nearly every proposition in which more than one truth is stated, the various parts are distinguished by numerals. Thus, when reference is made to a preceding section, the pupil will readily find the precise statement which is to be quoted.

The exercises are upwards of seven hundred in number, and have been selected with great care. In certain exercises which might otherwise present difficulties to the pupil, reference is made to a previous section or exercise which may be used in the solution. The exercises in each Book are numbered consecutively.

In the Plane Geometry, the new exercises are largely numerical; but in the Solid Geometry, there is a considerable increase in the number of both numerical exercises and original theorems. A number of the exercises are in the nature of alternative methods of proof for preceding propositions.

In the Appendix to the Plane Geometry will be found an additional set of exercises of somewhat greater difficulty than those previously given.

The pages have been arranged in such a way as to avoid the necessity, while reading a proof, of turning the page for reference to the figure.

The attention of teachers is specially invited to the explanations given in the Introduction, commencing on page vii.

The author desires to express his thanks to the many teachers, in all parts of the country, who have furnished him with valuable suggestions and criticisms.

WEBSTER WELLS.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
1894.

CONTENTS.

TO TEACHERS.

Among the most important objects of the study of Geometry are the development of the reasoning faculties, and the cultivation of the power of clear and accurate expression.

To attain these ends, the pupil should be required to state the various parts of a demonstration in concise and logical language, and to give after each statement of a proof *the reason in full.*

Throughout Book I., and in the first sixteen propositions of Book VI., the required authority is printed in each case directly after the statement, in smaller type, enclosed in brackets.

In the remaining portions of the work, the formal statement of the reason is omitted, and there is given, in parenthesis, only the number of the section where the authority is to be found.

In every such case, the pupil should be held, as in Book I., to the full statement of the reason.

The statements of the corollaries are in all cases printed in italics; so that when a previous section is referred to in a proof, the pupil will always find *printed in italics* the precise statement to be quoted.

Thus in Prop. II., Book II., reference is made to § 143; this calls for the following statement:—

All radii of a circle are equal.

While in general the complete statement of the reference should be insisted on, if the proposition referred to states more than one truth, that portion only need be quoted which applies to the case under consideration.

Thus, in the proof of § 29, reference is made to § 28; here the complete reference is not given, but only the portion actually used.

In most cases, the various parts of a proposition are indicated by numerals; and when reference is made to a section, the numeral following the number of the section shows which portion of the statement is to be quoted.

Thus, in Prop. XXI., Book II., Case I., reference is made to § 83, 1; this calls for the following statement: —

An exterior angle of a triangle is equal to the sum of the two opposite interior angles.

If a previous case of the same proposition is referred to, the reference given should be the statement of the theorem, followed by the statement of the previous case.

Thus, on page 92, the reference "§ 189, Case I." calls for the following: —

In the same circle, two central angles are in the same ratio as their intercepted arcs, when the arcs are commensurable.

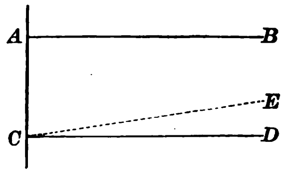
Considerable practice should be given in writing demonstrations on the blackboard; the authority for each statement being given in full, just as when the proof is given orally. The symbols given on page 4 should be used whenever possible. The following abbreviations will also be found of use: —

Ax.,	Axiom.	Sup.,	Supplementary.
Def.,	Definition.	Alt.,	Alternate.
Hyp.,	Hypothesis.	Int.,	Interior.
Cons.,	Construction.	Ext.,	Exterior.
Rt.,	Right.	Corresp.,	Corresponding.
Str.,	Straight.	Rect.,	Rectangle.
Adj.,	Adjacent.		

The author has thought that it would be an aid to teachers to put a few propositions in a form which is recommended for blackboard work.

Prop. XIII. Book I.

A straight line perpendicular to one of two parallels is perpendicular to the other.



Let the lines AB and CD be $\parallel$, and let AC be $\perp$ to AB.
To prove $AC \perp CD$.

If CD is not $\perp$ to AC, let CE be drawn $\perp$ to AC.

$$\therefore AB \parallel CE.$$

[Two $\perp$s to the same str. line are $\parallel$.]

But by hyp., $AB \parallel CD$.

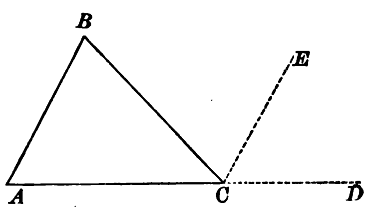
$\therefore$ CE must coincide with CD.

[But one str. line can be drawn through a given point $\parallel$ to a given str. line.]

But $AC \perp CE$, and $\therefore AC \perp CD$.

Prop. XXVI. Book I.

The sum of the angles of any triangle is equal to two right angles.



Let ABC be any $\triangle$.

To prove $\angle A + \angle B + \angle C =$ two rt. $\angle$s.

Produce AC to D, and draw $CE \parallel AB$.

Then, $\angle ECD = \angle A$.

[If two ‖s are cut by a secant line, the corresp. $\angle$s are equal.]

Again, $\angle BCE = \angle B$.

[If two ‖s are cut by a secant line, the alt.-int. $\angle$s are equal.]

But, $\angle ECD + \angle BCE + \angle ACB =$ two rt. $\angle$s.

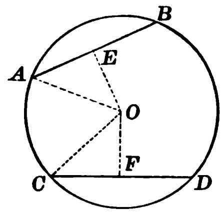
[The sum of all the $\angle$s formed on the same side of a str. line at a given point is equal to two rt. $\angle$s.]

Putting $\angle ECD = \angle A$, and $\angle BCE = \angle B$, we have

$$\angle A + \angle B + \angle ACB = \text{two rt. } \angle\text{s}.$$

PROP. X. BOOK II.

In the same circle, or in equal circles, equal chords are equally distant from the centre.



Let AB and CD be equal chords of the $\odot$ ABD.

To prove AB and CD equally distant from the centre O.

Draw OE and OF $\perp$ to AB and CD, respectively, and draw OA and OC.

Then E is the middle point of AB, and F of CD.

[The diameter $\perp$ to a chord bisects it.]

Now in the rt. $\triangle$s OAE and OCF,

$$AE = CF,$$

being halves of equal chords.

Also, $$OA = OC.$$

[All radii of a ⊙ are equal.]

$$\therefore \triangle OAE = \triangle OCF.$$

[Two rt. ⚠ are equal when the hypotenuse and a leg of one are equal respectively to the hypotenuse and a leg of the other.]

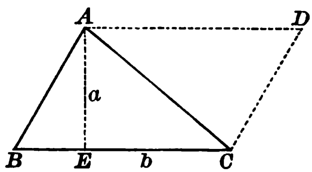
$$\therefore OE = OF.$$

[In equal figures, the homologous parts are equal.]

$\therefore AB$ and CD are equally distant from O.

PROP. V. BOOK IV.

The area of a triangle is equal to one-half the product of its base and altitude.



Let ABC be a $\triangle$, having its altitude $AE = a$, and its base $BC = b$.

To prove $$\text{area } ABC = \tfrac{1}{2}\, a \times b.$$

Draw AD and CD ∥ to BC and AB, respectively, forming the ▱ $ABCD$.

Now, $$\triangle ABC = \triangle ACD.$$

[A diagonal of a ▱ divides it into two equal ⚠.]

$$\therefore \text{area } ABC = \tfrac{1}{2} \text{ area } ABCD.$$

But, $$\text{area } ABCD = a \times b.$$

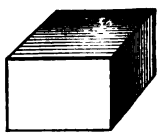
[The area of a ▱ is equal to the product of its base and altitude.]

$$\therefore \text{area } ABC = \tfrac{1}{2}\, a \times b.$$

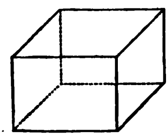
GEOMETRY.

NOTE. The attention of teachers is particularly called to the explanations given in the Introduction, commencing on p. vii.

PRELIMINARY DEFINITIONS.



A material body.



A geometrical solid.

1. A *material body*, such as a block of wood, occupies a *limited* or *bounded* portion of space.

The boundary which separates such a body from surrounding space is called the *surface* of the body.

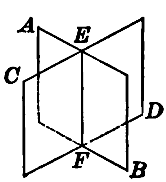
2. If the material composing such a body could be conceived as taken away from it, *without altering the form or shape of the bounding surface*, there would remain a *portion of space* having the same bounding surface as the former material body.

This is called a *geometrical solid*, or simply a *solid*.

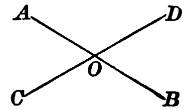
The bounding surface is called the *surface* of the solid.

3. If two surfaces cut each other, their common intersection is called a *line*.

Thus, if the surfaces AB and CD cut each other, their common intersection, EF, is a line.



4. If two lines cut each other, their common intersection is called a *point*.



Thus, if the lines AB and CD cut each other, their common intersection, O, is a point.

5. A solid has *extension in every direction;* but this is not true of surfaces and lines.

A point has extension in *no* direction, but simply *position in space.*

6. A surface may be conceived as existing independently in space, without reference to the solid whose boundary it forms.

In like manner, we may conceive of lines and points as having an independent existence in space.

7. A *straight line* is a line which has the same direction throughout its length; as AB.

A straight line is also called a *right line.*

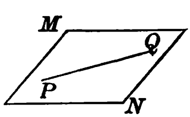
NOTE. The word "*line*" will be used hereafter as signifying a *straight line.*

A straight line. *A curve.* *A broken line.*

A *curved line,* or simply a *curve,* is a line no portion of which is straight; as CD.

A *broken line* is a line which is composed of different successive straight lines; as $EFGH$.

8. A *plane surface,* or simply a *plane,* is a surface such that the straight line joining any two of its points lies entirely in the surface.



Thus, the surface MN is a plane if the straight line PQ joining any two of its points lies entirely in the surface.

9. A *curved surface* is a surface no portion of which is plane.

10. We may conceive of a straight line as being of unlimited extent in regard to length; and in like manner we may conceive of a plane as being of unlimited extent in regard to length and breadth.

11. A *geometrical figure* is any combination of points, lines, surfaces, or solids.

12. A *plane figure* is a figure formed by points and lines all lying in the same plane.

13. A geometrical figure is called *rectilinear*, or *right-lined*, when it is composed of straight lines only.

14. *Geometry* treats of the properties, construction, and measurement of geometrical figures.

15. *Plane Geometry* treats of plane figures only; *Solid Geometry*, also called *Geometry of Space*, or *Geometry of Three Dimensions*, treats of figures which are not plane.

16. An *Axiom* is a truth assumed as self-evident.

A *Theorem* is a truth which requires demonstration.

A *Problem* is a question proposed for solution.

A *Proposition* is a general term for either a theorem or a problem.

A *Postulate* is an assumption of the possibility of solving a certain problem.

A *Corollary* is a secondary theorem, which is an immediate consequence of the proposition which it follows.

A *Scholium* is a remark or note.

An *Hypothesis* is a supposition made either in the statement or the demonstration of a proposition.

One proposition is said to be the *Converse* of another when the hypothesis and conclusion of the first are respectively the conclusion and hypothesis of the second.

17. POSTULATES.

1. A straight line can be drawn between any two points.

2. A straight line can be produced indefinitely in either direction.

18. AXIOMS.

1. *Things which are equal to the same thing, or to equals, are equal to each other.*

2. *If the same operation be performed upon equals, the results will be equal.*

3. *The whole is equal to the sum of all its parts.*

4. *The whole is greater than any of its parts.*

5. *But one straight line can be drawn between two points.*

6. *A straight line is the shortest line between two points.*

19. A straight line is said to be *determined* by any two of its points.

SYMBOLS.

20. The following symbols will be used in the work:

$\times$, multiplied by.	$\angle$, angle.
$-$, minus.	$<$, is less than.
$+$, plus.	$>$, is greater than.
$=$, equals.	$\triangle$, triangle.
$\backsimeq$, is equivalent to.	

In addition to these, the following are useful in writing demonstrations on the blackboard, or in exercise books:

$\angle$s, angles.	$\square$, parallelogram.
$\triangle$s, triangles.	$\square$s, parallelograms.
$\perp$, perpendicular.	$\odot$, circle.
$\perp$s, perpendiculars.	Ⓢ, circles.
$\parallel$, parallel.	$\therefore$, therefore.
$\parallel$s, parallels.	

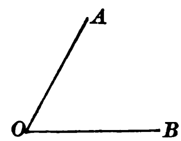
PLANE GEOMETRY.

BOOK I.

RECTILINEAR FIGURES.

DEFINITIONS AND GENERAL PRINCIPLES.

21. If two straight lines be drawn from the same point in different directions, the figure formed is called an *Angle*.



Thus, if the straight lines OA and OB be drawn from the same point O in different directions, the figure AOB is an angle.

The point O is called the *vertex* of the angle, and the lines OA and OB are called its *sides*.

The symbol $\angle$ is used for the word "angle."

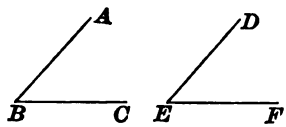
22. If there is but one angle at a given vertex, it may be designated by the letter at that vertex; but if two or more angles have the same vertex, it is necessary, in order to avoid ambiguity, to name also the letters at the extremities of the sides, placing the letter at the vertex between the others.

Thus, we should call the angle of the preceding article "the angle O"; but if there were other angles having the same vertex, we should read it either AOB or BOA.

Another method of designating an angle is by means of a letter placed between its sides; an example of this will be found in § 71.

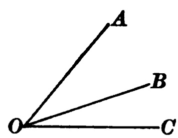
23. Two geometrical figures are said to be *equal* when one can be applied to the other so that they shall coincide throughout.

24. To prove that two *angles* are equal, it is not necessary to consider the lengths of their sides.

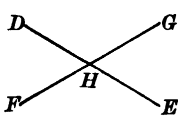


Thus, if the angle ABC can be applied to the angle DEF so that the point B shall fall on E, and the sides AB and BC on DE and EF, respectively, the angles are equal, even if the sides AB and BC are not equal in length to DE and EF, respectively.

25. Two angles are called *adjacent* when they have the same vertex, and a common side between them; as AOB and BOC.



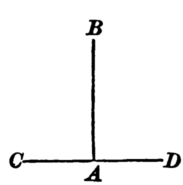
26. Two angles are called *vertical*, or *opposite*, when the sides of one are the prolongations of the sides of the other; as DHF and GHE.



PERPENDICULAR LINES.

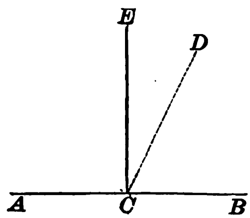
27. If through a given point in a straight line a line be drawn meeting the given line in such a way as to make the adjacent angles equal, each of the equal angles is called a *right angle*, and the lines are said to be *perpendicular* to each other.

Thus, if A be any point in the line CD, and the line AB be drawn in such a way as to make the angles BAC and BAD equal, each of these angles is a right angle, and the lines AB and CD are perpendicular to each other.



Proposition I. Theorem.

28. *At a given point in a straight line, a perpendicular to the line can be drawn, and but one.*



Let C be the given point in the straight line AB.

To prove that a perpendicular can be drawn to AB at C, and but one.

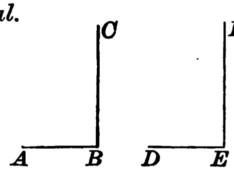
Draw CD, making $\angle BCD < \angle ACD$; and let CD be revolved about the point C as a pivot towards the position CA.

Then, $\angle BCD$ will constantly increase, and $\angle ACD$ will constantly diminish; and there must be one position of CD, and only one, where the angles are equal.

Let CE be this position; then CE is perpendicular to AB.

Hence, a perpendicular can be drawn to AB at C, and but one.

29. Cor. *All right angles are equal.*



Let ABC and DEF be right angles.

To prove $\angle ABC = \angle DEF$.

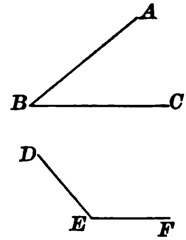
Apply $\angle ABC$ to $\angle DEF$ in such a way that AB shall fall upon DE; the point B falling at E.

Then BC will fall upon EF; for otherwise we should have two perpendiculars to DE at E, which is impossible.

[At a given point in a straight line, but one perpendicular to the line can be drawn.] (§ 28.)

Whence, $\angle ABC = \angle DEF$.

DEFINITIONS.



30. An *acute* angle is an angle which is less than a right angle; as *ABC*.

An *obtuse* angle is an angle which is greater than a right angle; as *DEF*.

Acute and obtuse angles are called *oblique* angles; and intersecting lines which are not perpendicular, are said to be oblique to each other.

31. An angle is *measured* by finding how many times it contains another angle, adopted arbitrarily as the unit of measure.

The usual unit of measure is the *degree*, which is the ninetieth part of a right angle.

To express fractional parts of the unit, the degree is divided into sixty equal parts, called *minutes*, and the minute into sixty equal parts, called *seconds*.

Degrees, minutes, and seconds are represented by the symbols, $^\circ$, $'$, $''$, respectively.

Thus, $43^\circ\ 22'\ 37''$ represents an angle of 43 degrees, 22 minutes, and 37 seconds.

32. If the sum of two angles is a right angle, or 90°, one is called the *complement* of the other; and if their sum is two right angles, or 180°, one is called the *supplement* of the other.

Thus, the complement of an angle of 34° is $90^\circ - 34^\circ$, or 56°; and its supplement is $180^\circ - 34^\circ$, or 146°.

Two angles which are complements of each other are called *complementary;* and two angles which are supplements of each other are called *supplementary*.

33. It is evident that

1. *The complements of equal angles are equal.*
2. *The supplements of equal angles are equal.*

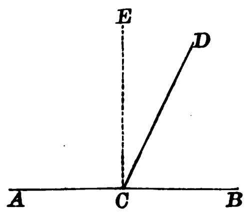
EXERCISES.

1. How many degrees are there in the complement of 47°? of 83°? of 90°?

2. How many degrees are there in the supplement of 31°? of 90°? of 178°?

PROPOSITION II. THEOREM.

34. *If one straight line meet another, the sum of the adjacent angles formed is equal to two right angles.*



Let the straight line CD meet the straight line AB at C.

To prove $\angle ACD + \angle BCD =$ two right angles.

Draw CE perpendicular to AB at C.

[At a given point in a straight line, a perpendicular to the line can be drawn.] (§ 28.)

Then, $\angle ACD + \angle BCD = \angle ACE + \angle BCE$.

But each of the angles ACE and BCE is a right angle.

Hence, $\angle ACD + \angle BCD =$ two right angles.

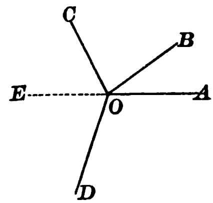
35. SCH. Since each of the angles ACD and BCD is the *supplement* of the other (§ 32), the theorem may be stated as follows:

If one straight line meet another, each of the adjacent angles formed is the supplement of the other.

Such angles are called *supplementary-adjacent.*

36. COR. I. *The sum of all the angles formed on the same side of a straight line at a given point is equal to two right angles.*

37. Cor. II. *The sum of all the angles formed about a point in a plane is equal to four right angles.*



Let AOB, BOC, COD, and DOA be angles formed about the point O.

To prove that the sum of the angles AOB, BOC, COD, and DOA is equal to four right angles.

Produce AO to E.

Then, the sum of the angles AOB, BOC, and COE is equal to two right angles.

[The sum of all the angles formed on the same side of a straight line at a given point is equal to two right angles.] (§ 36.)

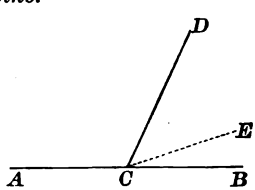
In like manner, the sum of the angles EOD and DOA is equal to two right angles.

Therefore, the sum of the angles AOB, BOC, COD, and DOA is equal to four right angles.

Ex. 3. If in the above figure the angles AOB, BOC, and COD are respectively 49°, 88°, and $\frac{9}{8}$ of a right angle, how many degrees are there in AOD?

Proposition III. Theorem.

38. (Converse of Prop. II.) *If the sum of two adjacent angles is equal to two right angles, their exterior sides lie in the same straight line.*



Let the sum of the adjacent angles ACD and BCD be equal to two right angles.

To prove that AC and BC lie in the same straight line.

Let CE be in the same straight line with AC.

Then, $\angle ECD$ is the supplement of $\angle ACD$.

[If one straight line meet another, each of the adjacent angles formed is the supplement of the other.] (§ 35.)

But by hypothesis,

$$\angle ACD + \angle BCD = \text{two right angles.}$$

Whence, $\angle BCD$ is the supplement of $\angle ACD$.

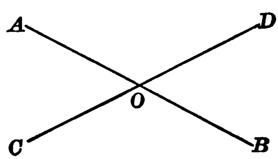
Therefore, $\angle ECD = \angle BCD$.

[The supplements of equal angles are equal.] (§ 33.)

Hence, EC coincides with BC, and AC and BC lie in the same straight line.

Proposition IV. Theorem.

39. *If two straight lines intersect, the vertical angles are equal.*



Let the straight lines AB and CD intersect at O.

To prove $\angle AOC = \angle BOD$.

$\angle AOC$ is the supplement of $\angle AOD$.

[If one straight line meet another, each of the adjacent angles formed is the supplement of the other.] (§ 35.)

In like manner, $\angle BOD$ is the supplement of $\angle AOD$.

Therefore, $\angle AOC = \angle BOD$.

[The supplements of equal angles are equal.] (§ 33.)

In like manner, we may prove

$$\angle AOD = \angle BOC.$$

Ex. 4. If in the above figure $\angle AOD = 137°$, how many degrees are there in BOC? in AOC? in BOD?

PROPOSITION V. THEOREM.

40. *If a perpendicular be erected at the middle point of a straight line,*

I. *Any point in the perpendicular is equally distant from the extremities of the line.*

II. *Any point without the perpendicular is unequally distant from the extremities of the line.*

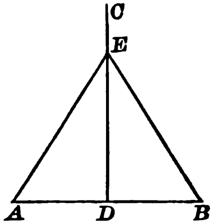


Fig. 1.

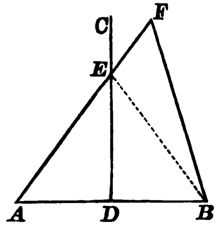


Fig. 2.

I. Let CD (Fig. 1) be perpendicular to AB at its middle point, D.

Let E be any point in CD, and draw AE and BE.

To prove $AE = BE$.

Let the figure BDE be superposed upon ADE by folding it over about DE as an axis.

We have, $\angle BDE = \angle ADE$.

[All right angles are equal.] (§ 29.)

Then the line BD will fall upon AD; and since, by hypothesis, $BD = AD$, the point B will fall at A.

Then the line BE will coincide with AE.

[But one straight line can be drawn between two points.] (Ax. 5.)

Therefore, $AE = BE$.

II. Let CD (Fig. 2) be perpendicular to AB at its middle point, D.

Let F be any point without CD, and draw AF and BF.

To prove $AF > BF$.

Let AF intersect CD at E, and draw BE.

Then, $BE + EF > BF$.

[A straight line is the shortest line between two points.] (Ax. 6.)

But, $BE = AE$.

[If a perpendicular be erected at the middle point of a straight line, any point in the perpendicular is equally distant from the extremities of the line.] (§ 40, I.)

Substituting AE for its equal BE, we have,

$$AE + EF > BF,$$

or, $$AF > BF.$$

41. Cor. I. When the figure BDE is superposed upon ADE, in the proof of § 40, I., $\angle EBD$ coincides with $\angle EAD$, and $\angle BED$ with $\angle AED$.

That is, $\angle EAD = \angle EBD$, and $\angle AED = \angle BED$; therefore,

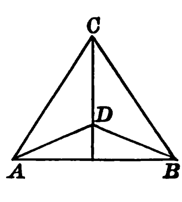
If lines be drawn to the extremities of a straight line from any point in the perpendicular erected at its middle point,

1. *They make equal angles with the line.*
2. *They make equal angles with the perpendicular.*

42. Cor. II. *Every point which is equally distant from the extremities of a straight line, lies in the perpendicular erected at the middle point of the line.*

43. Cor. III. A straight line is determined by any two of its points (§ 19); hence, by § 42,

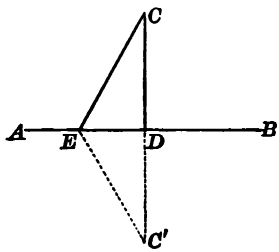
Two points, each equally distant from the extremities of a straight line, determine a perpendicular at its middle point.



Thus, if each of the points C and D is equally distant from A and B, the line CD is perpendicular to AB at its middle point.

Proposition VI. Theorem.

44. ***From a given point without a straight line, but one perpendicular can be drawn to the line.***



Let C be the given point without the line AB, and draw CD perpendicular to AB.

To prove that CD is the only perpendicular that can be drawn from C to AB.

If possible, let CE be another perpendicular from C to AB.

Produce CD to C', making $C'D = CD$, and draw EC'.

Then since ED is perpendicular to CC' at its middle point D,

$$\angle\, CED = \angle\, C'ED.$$

[If lines be drawn to the extremities of a straight line from any point in the perpendicular erected at its middle point, they make equal angles with the perpendicular.] (§ 41.)

But by hypothesis, $\angle\, CED$ is a right angle.

Then its equal, $\angle\, C'ED$, is also a right angle.

Whence, $\angle\, CED + \angle\, C'ED =$ two right angles.

Therefore, CEC' is a straight line.

[If the sum of two adjacent angles is equal to two right angles, their exterior sides lie in the same straight line.] (§ 38.)

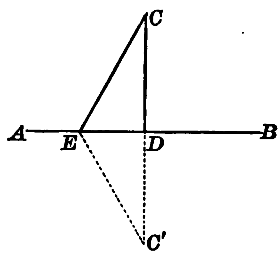
But this is impossible, since, by construction, CDC' is a straight line.

[But one straight line can be drawn between two points.] (Ax. 5.)

Hence, CE cannot be perpendicular to AB, and CD is the only perpendicular that can be drawn.

Proposition VII. Theorem.

45. *The perpendicular is the shortest line that can be drawn from a point to a straight line.*



Let CD be the perpendicular from C to the line AB, and CE any other line drawn from C to AB.

To prove $$CD < CE.$$

Produce CD to C', making $C'D = CD$, and draw EC'.

Then since ED is perpendicular to CC' at its middle point D,

$$CE = C'E.$$

[If a perpendicular be erected at the middle point of a straight line, any point in the perpendicular is equally distant from the extremities of the line.] (§ 40.)

But $$CD + DC' < CE + EC'.$$

[A straight line is the shortest line between two points.] (Ax. 6.)

Therefore, $$2\,CD < 2\,CE,$$

or $$CD < CE.$$

46. Sch. The *distance* of a point from a line is understood to mean the length of the perpendicular from the point to the line.

EXERCISES.

5. How many degrees are there in the complement, and in the supplement, of an angle equal to $\frac{7}{12}$ of a right angle?

6. How many degrees are there in an angle whose supplement is equal to $\frac{26}{11}$ of its complement?

7. Two angles are complementary, and the greater exceeds the less by 37°. How many degrees are there in each angle?

8. Two angles are supplementary, and the greater is seven times the less. How many degrees are there in each angle?

9. Find the number of degrees in the angle the sum of whose supplement and complement is 196°.

10. The straight line which bisects an angle bisects also its vertical angle. (§ 39.)

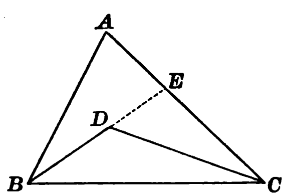
11. The bisectors of a pair of vertical angles lie in the same straight line.

12. The bisectors of two supplementary adjacent angles are perpendicular to each other.

13. The line passing through the vertex of an angle perpendicular to its bisector bisects the supplementary adjacent angle.

PROPOSITION VIII. THEOREM.

47. *If two lines be drawn from a point to the extremities of a straight line, their sum is greater than the sum of two other lines similarly drawn, but enveloped by them.*



Let the lines AB and AC be drawn from the point A to the extremities of the line BC; and let DB and DC be two other lines similarly drawn, but enveloped by AB and AC.

To prove $AB + AC > BD + DC$.

Produce BD to meet AC at E.

Then, $$BA + AE > BE.$$

[A straight line is the shortest line between two points.] (Ax. 6.)

Whence, the broken line BAC is greater than BEC.

In like manner, $$DE + EC > DC.$$

Whence, the broken line BEC is greater than BDC.

Therefore, the broken line BAC is greater than BDC.

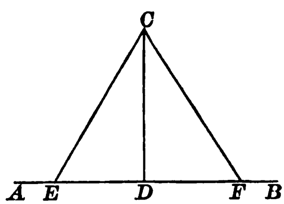
That is, $$AB + AC > DB + DC.$$

Proposition IX. Theorem.

48. *If oblique lines be drawn from a point to a straight line,*

I. *Two oblique lines cutting off equal distances from the foot of the perpendicular from the point to the line are equal.*

II. *Of two oblique lines cutting off unequal distances from the foot of the perpendicular, the more remote is the greater.*



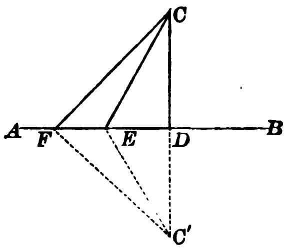
I. Let CD be the perpendicular from C to AB.

Let CE and CF be oblique lines drawn from C to AB, cutting off equal distances from the foot of the perpendicular.

To prove $$CE = CF.$$

Since CD is perpendicular to EF at its middle point D,
$$CE = CF.$$

[If a perpendicular be erected at the middle point of a straight line, any point in the perpendicular is equally distant from the extremities of the line.] (§ 40.)



II. Let CD be the perpendicular from C to AB.

Let CE and CF be oblique lines drawn from C to AB, cutting off unequal distances from the foot of the perpendicular, CF being the more remote.

To prove $$CF > CE.$$

Produce CD to C', making $C'D = CD$, and draw $C'E$ and $C'F$.

Then since AD is perpendicular to CC' at its middle point D,

$$CF = C'F, \text{ and } CE = C'E.$$

[If a perpendicular be erected at the middle point of a straight line, any point in the perpendicular is equally distant from the extremities of the line.] (§ 40.)

But, $$CF + FC' > CE + EC'.$$

[If two lines be drawn from a point to the extremities of a straight line, their sum is greater than the sum of two other lines similarly drawn, but enveloped by them.] (§ 47.)

Therefore, $2\ CF > 2\ CE$, or $CF > CE$.

NOTE. The theorem holds equally if the oblique line CE is on the opposite side of the perpendicular CD from CF.

49. COR. *But two equal oblique lines can be drawn from a point to a straight line.*

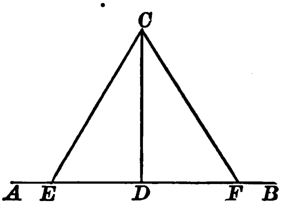
EXERCISES.

14. If the bisectors of two adjacent angles are perpendicular, the angles are supplementary.

15. A line drawn through the vertex of an angle perpendicular to its bisector makes equal angles with the sides of the given angle.

PROPOSITION X. THEOREM.

50. (Converse of Prop. IX., I.) *If oblique lines be drawn from a point to a straight line, two equal oblique lines cut off equal distances from the foot of the perpendicular from the point to the line.*



Let CD be the perpendicular from C to AB.

Let CE and CF be equal oblique lines drawn from C to AB.

To prove $DE = DF$.

If DE were greater than DF, CE would be greater than CF.

[If oblique lines be drawn from a point to a straight line, of two oblique lines cutting off unequal distances from the foot of the perpendicular, the more remote is the greater.] (§ 48.)

And if DE were less than DF, CE would be less than CF.

But each of these conclusions is contrary to the hypothesis that $CE = CF$.

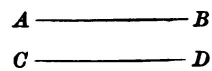
Therefore, $DE = DF$.

NOTE. The method of proof exemplified in the above proposition is known as the "Indirect Method," or the "*Reductio ad Absurdum*"; the truth of a theorem is proved by supposing it to be false, and showing that the hypothesis leads to a false conclusion.

51. COR. (Converse of Prop. IX., II.) *If two unequal oblique lines be drawn from a point to a straight line, the greater cuts off the greater distance from the foot of the perpendicular from the point to the line.*

PARALLEL LINES.

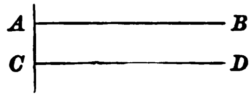
52. Definition. Two straight lines are said to be *parallel* when they lie in the same plane, and cannot meet however far they may be produced; as AB and CD.



53. Axiom. *But one straight line can be drawn through a given point parallel to a given straight line.*

Proposition XI. Theorem.

54. *Two perpendiculars to the same straight line are parallel.*



Let the lines AB and CD be perpendicular to AC.

To prove AB and CD parallel.

If AB and CD are not parallel, they will meet in some point if sufficiently produced (§ 52).

We should then have two perpendiculars from the same point to AC, which is impossible.

[From a given point without a straight line, but one perpendicular can be drawn to the line.] (§ 44.)

Therefore, AB and CD cannot meet, and are parallel.

Proposition XII. Theorem.

55. *Two straight lines parallel to the same straight line are parallel to each other.*

Let the lines AB and CD be parallel to EF.

To prove AB and CD parallel to each other.

If AB and CD are not parallel, they will meet in some point if sufficiently produced (§ 52).

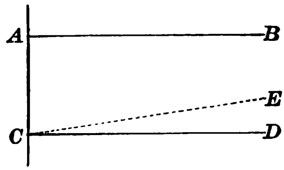
We should then have two lines drawn through the same point parallel to EF, which is impossible.

[But one straight line can be drawn through a given point parallel to a given straight line.] (§ 53.)

Therefore, AB and CD cannot meet, and are parallel.

Proposition XIII. Theorem.

56. *A straight line perpendicular to one of two parallels is perpendicular to the other.*



Let the lines AB and CD be parallel, and let AC be perpendicular to AB.

To prove AC perpendicular to CD.

If CD is not perpendicular to AC, let CE be drawn perpendicular to AC.

Then AB and CE are parallel.

[Two perpendiculars to the same straight line are parallel.] (§ 54.)

But, by hypothesis, AB and CD are parallel.

Then CE must coincide with CD.

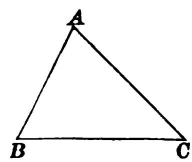
[But one straight line can be drawn through a given point parallel to a given straight line.] (§ 53.)

But AC is perpendicular to CE, and therefore AC is perpendicular to CD.

TRIANGLES.

Definitions.

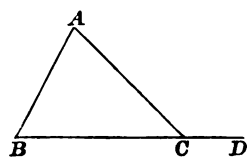
57. A *triangle* is a portion of a plane bounded by three straight lines; as *ABC*.



The bounding lines, *AB*, *BC*, and *CA*, are called the *sides* of the triangle, and their points of intersection, *A*, *B*, and *C*, are called the *vertices*.

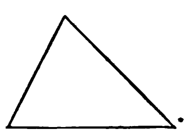
The *angles* of the triangle are the angles *CAB*, *ABC*, and *BCA*, formed by the adjacent sides.

An *exterior angle* of a triangle is the angle formed at any vertex by any side of the triangle and the adjacent side produced; as *ACD*.

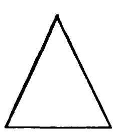


The symbol △ is used for the word "triangle."

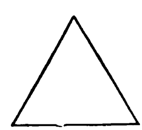
58. A triangle is called *scalene* when no two of its sides are equal; *isosceles* when two of its sides are equal; *equilateral* when all its sides are equal; and *equiangular* when all its angles are equal.



Scalene.

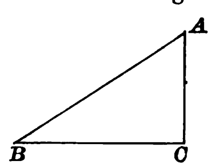


Isosceles.



Equilateral.

59. A *right triangle* is a triangle which has a right angle; as *ABC*, which has a right angle at *C*.

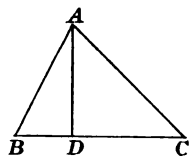


The side *AB* opposite to the right angle is called the *hypotenuse*, and the other sides, *AC* and *BC*, the *legs*.

60. The *base* of a triangle is the side upon which it is supposed to stand.

In general, any side may be taken as the base; but in an isosceles triangle, unless otherwise specified, the side which is not one of the equal sides is taken as the base.

When any side has been taken as the base, the opposite angle is called the *vertical angle*, and its vertex is called the vertex of the triangle.



The *altitude* of a triangle is the perpendicular drawn from the vertex to the base, produced if necessary.

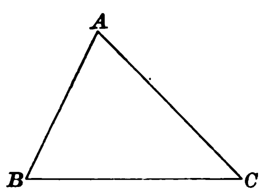
Thus, in the triangle ABC, BC is the base, BAC the vertical angle, and AD the altitude.

61. Since a straight line is the shortest line between two points (Ax. 6), it follows that

Either side of a triangle is less than the sum of the other two sides.

PROPOSITION XIV. THEOREM.

62. *Either side of a triangle is greater than the difference of the other two sides.*



In the triangle ABC, let BC be greater than AC.

To prove $AB > BC - AC$.

We have $AB + AC > BC$.

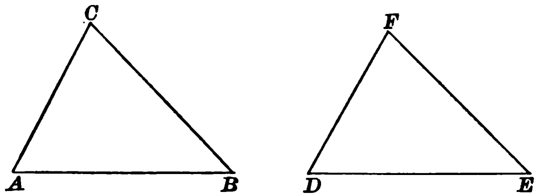
[A straight line is the shortest line between two points.] (Ax. 6.)

Subtracting AC from both members of the inequality,

$$AB > BC - AC.$$

Proposition XV. Theorem.

63. *Two triangles are equal when two sides and the included angle of one are equal respectively to two sides and the included angle of the other.*



In the triangles ABC and DEF, let

$$AB = DE,\ AC = DF, \text{ and } \angle A = \angle D.$$

To prove $\triangle ABC = \triangle DEF.$

Superpose the triangle ABC upon DEF in such a way that $\angle A$ shall coincide with its equal $\angle D$; the side AB falling upon DE, and the side AC upon DF.

Then since $AB = DE$ and $AC = DF$, the point B will fall at E, and the point C at F.

Whence, the side BC will coincide with EF.

[But one straight line can be drawn between two points.] (Ax. 5.)

Therefore, ABC and DEF coincide throughout, and are equal.

64. Cor. Since ABC and DEF coincide throughout, we have

$$\angle B = \angle E,\ \angle C = \angle F, \text{ and } BC = EF.$$

65. Sch. I. In equal figures, lines or angles which are similarly placed are called *homologous.*

Thus, in the figure of Prop. XV., $\angle A$ is homologous to $\angle D$; AB is homologous to DE; etc.

66. Sch. II. It follows from § 65 that

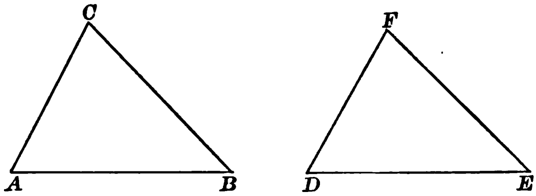
In equal figures, the homologous parts are equal.

67. SCH. III. In equal triangles, the equal angles lie opposite the equal sides.

Ex. 16. If, in the figure of Prop. XV., $AB = EF$, $BC = DE$, and $\angle B = \angle E$, which angle of the triangle DEF is equal to A? which angle is equal to C?

PROPOSITION XVI. THEOREM.

68. *Two triangles are equal when a side and two adjacent angles of one are equal respectively to a side and two adjacent angles of the other.*



In the triangles ABC and DEF, let

$$AB = DE, \ \angle A = \angle D, \text{ and } \angle B = \angle E.$$

To prove $\triangle ABC = \triangle DEF$.

Superpose the triangle ABC upon DEF in such a way that the side AB shall coincide with its equal DE; the point A falling at D, and the point B at E.

Then since $\angle A = \angle D$, the side AC will fall upon DF, and the point C will fall somewhere in DF.

And since $\angle B = \angle E$, the side BC will fall upon EF, and the point C will fall somewhere in EF.

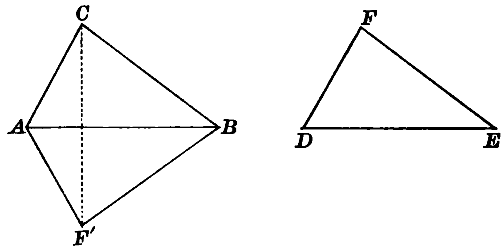
Then the point C, falling at the same time in DF and EF, must fall at their intersection F.

Therefore, ABC and DEF coincide throughout, and are equal.

Ex. 17. If, in the figure of Prop. XVI., $AC = DF$, $\angle A = \angle F$, and $\angle C = \angle D$, which side of the triangle DEF is equal to AB? which side is equal to BC?

PROPOSITION XVII. THEOREM.

69. *Two triangles are equal when the three sides of one are equal respectively to the three sides of the other.*



In the triangles ABC and DEF, let

$$AB = DE,\ BC = EF, \text{ and } CA = FD.$$

To prove $\triangle ABC = \triangle DEF.$

Place the triangle DEF in the position ABF'; the side DE coinciding with its equal AB, and the vertex F falling at F', on the opposite side of AB from C.

Draw CF'.

Then since, by hypothesis, $AC = AF'$ and $BC = BF'$, AB is perpendicular to CF' at its middle point.

[Two points, each equally distant from the extremities of a straight line, determine the perpendicular at its middle point.] (§ 43.)

Whence, $\angle BAC = \angle BAF'.$

[If lines be drawn to the extremities of a straight line from any point in the perpendicular erected at its middle point, they make equal angles with the perpendicular.] (§ 41.)

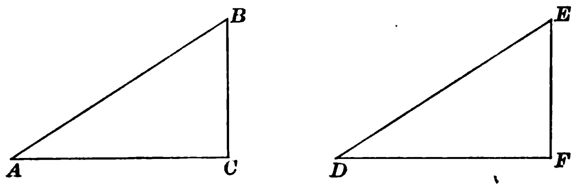
Therefore, $\triangle ABC = \triangle ABF'.$

[Two triangles are equal when two sides and the included angle of one are equal respectively to two sides and the included angle of the other.] (§ 63.)

That is, $\triangle ABC = \triangle DEF.$

Proposition XVIII. Theorem.

70. *Two right triangles are equal when the hypotenuse and an adjacent angle of one are equal respectively to the hypotenuse and an adjacent angle of the other.*



In the right triangles ABC and DEF, let the hypotenuse AB be equal to DE, and $\angle A = \angle D$.

To prove $\triangle ABC = \triangle DEF$.

Superpose the triangle ABC upon DEF in such a way that the hypotenuse AB shall coincide with its equal DE; the point A falling at D, and the point B at E.

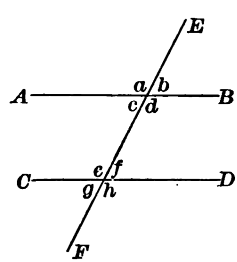
Then since $\angle A = \angle D$, the side AC will fall upon DF.

Whence, the side BC will fall upon EF.

[From a given point without a straight line, but one perpendicular can be drawn to the line.] (§ 44.)

Hence, ABC and DEF coincide throughout, and are equal.

71. Def. If two straight lines, AB and CD, are cut by a line EF, called a *secant line*, the angles formed are named as follows:—



c, d, e, and f are called *interior* angles, and a, b, g, and h *exterior* angles.

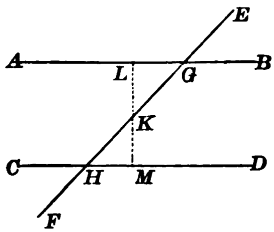
c and f, or d and e, are called *alternate-interior* angles.

a and h, or b and g, are called *alternate-exterior* angles.

a and e, b and f, c and g, or d and h, are called *corresponding* angles.

Proposition XIX. Theorem.

72. *If two parallels are cut by a secant line, the alternate-interior angles are equal.*



Let the parallels AB and CD be cut by the secant line EF in the points G and H, respectively.

To prove $\angle AGH = \angle GHD$, and $\angle BGH = \angle CHG$.

Through K, the middle point of GH, draw LM perpendicular to AB.

Then LM is also perpendicular to CD.

[A straight line perpendicular to one of two parallels is perpendicular to the other.] (§ 56.)

Then in the right triangles GKL and HKM, we have

$$GK = HK.$$

Also, $\angle GKL = \angle HKM$.

[If two straight lines intersect, the vertical angles are equal.] (§ 39.)

Hence, $\triangle GKL = \triangle HKM$.

[Two right triangles are equal when the hypotenuse and an adjacent angle of one are equal respectively to the hypotenuse and an adjacent angle of the other.] (§ 70.)

Therefore, $\angle KGL = \angle KHM$.

[In equal figures, the homologous parts are equal.] (§ 66.)

Again, $\angle AGH$ is the supplement of $\angle BGH$, and $\angle GHD$ is the supplement of $\angle CHG$.

[If one straight line meet another, each of the adjacent angles formed is the supplement of the other.] (§ 35.)

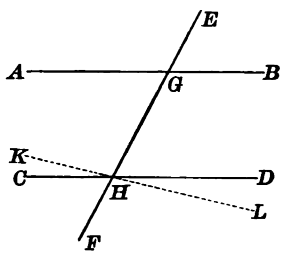
But, $\angle AGH = \angle GHD.$

Whence, $\angle BGH = \angle CHG.$

[The supplements of equal angles are equal.] (§ 33.)

Proposition XX. Theorem.

73. (Converse of Prop. XIX.) *If two straight lines are cut by a secant line, making the alternate-interior angles equal, the two lines are parallel.*



Let the lines AB and CD be cut by the secant line EF in the points G and H, respectively, making

$$\angle AGH = \angle GHD.$$

To prove AB and CD parallel.

Through H draw KL parallel to AB.

Then since the parallels AB and KL are cut by EF,

$$\angle AGH = \angle GHL.$$

[If two parallels are cut by a secant line, the alternate-interior angles are equal.] (§ 72.)

But by hypothesis, $\angle AGH = \angle GHD.$

Whence, $\angle GHL = \angle GHD.$

[Things which are equal to the same thing are equal to each other.] (Ax. 1.)

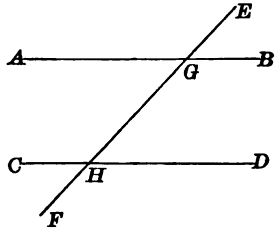
But this is impossible unless KL coincides with CD.

Therefore, CD is parallel to AB.

In like manner, it may be proved that if AB and CD are cut by EF, making $\angle BGH = \angle CHG$, then AB and CD are parallel.

PROPOSITION XXI. THEOREM.

74. *If two parallels are cut by a secant line, the corresponding angles are equal.*



Let the parallels AB and CD be cut by the secant line EF in the points G and H, respectively.

To prove $\angle AGE = \angle CHG.$

We have, $\angle BGH = \angle CHG.$

[If two parallels are cut by a secant line, the alternate-interior angles are equal.] (§ 72.)

But, $\angle BGH = \angle AGE.$

[If two straight lines intersect, the vertical angles are equal.] (§ 39.)

Whence, $\angle AGE = \angle CHG.$

[Things which are equal to the same thing are equal to each other.] (Ax. 1.)

In like manner, we may prove

$$\angle AGH = \angle CHF,\ \angle BGE = \angle DHG, \text{ and}$$
$$\angle BGH = \angle DHF.$$

75. COR. *If two parallels are cut by a secant line,*

I. *The alternate-exterior angles are equal.*

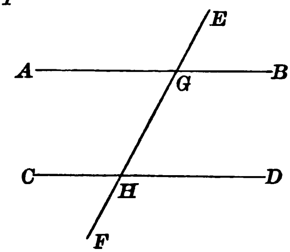
For example, $\angle AGE = \angle DHF.$

II. *The sum of the interior angles on the same side of the secant line is equal to two right angles.*

Thus, $\angle AGH + \angle CHG =$ two right angles.

PROPOSITION XXII. THEOREM.

76. (Converse of Prop. XXI.) *If two straight lines are cut by a secant line, making the corresponding angles equal, the two lines are parallel.*



Let the lines AB and CD be cut by the secant line EF in the points G and H, respectively, making

$$\angle AGE = \angle CHG.$$

To prove AB and CD parallel.

We have, $\angle AGE = \angle BGH.$

[If two straight lines intersect, the vertical angles are equal.] (§ 39.)

Whence, $\angle BGH = \angle CHG.$

Therefore, AB and CD are parallel.

[If two straight lines are cut by a secant line, making the alternate-interior angles equal, the two lines are parallel.] (§ 73.)

In like manner, it may be proved that, if

$$\angle AGH = \angle CHF, \text{ or } \angle BGE = \angle DHG, \text{ or}$$
$$\angle BGH = \angle DHF,$$

then AB and CD are parallel.

77. COR. (Converse of § 75.) *If two straight lines are cut by a secant line, making*

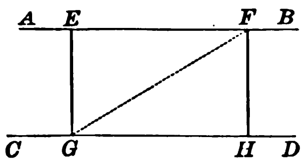
I. *The alternate-exterior angles equal;*

II. *The sum of the interior angles on the same side of the secant line equal to two right angles;*

the two lines are parallel.

Proposition XXIII. Theorem.

78. *Two parallel lines are everywhere equally distant.*



Let AB and CD be parallel lines, and E and F any two points on AB.

To prove E and F equally distant from CD.

Draw EG and FH perpendicular to CD.

Also, draw FG.

Now EG is perpendicular to AB.

[A straight line perpendicular to one of two parallels is perpendicular to the other.] (§ 56.)

Then in the right triangles EFG and FGH, FG is common.

And since the parallels AB and CD are cut by FG,

$$\angle EFG = \angle FGH.$$

[If two parallels are cut by a secant line, the alternate-interior angles are equal.] (§ 72.)

Hence, $\triangle EFG = \triangle FGH.$

[Two right triangles are equal when the hypotenuse and an adjacent angle of one are equal respectively to the hypotenuse and an adjacent angle of the other.] (§ 70.)

Therefore, $EG = FH.$

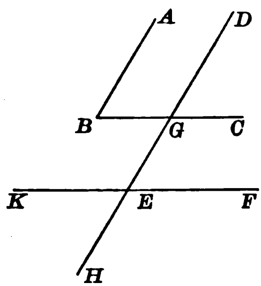
[In equal figures, the homologous parts are equal.] (§ 66.)

Hence, E and F are equally distant (§ 46) from CD.

Ex. 18. If, in the figure of Prop. XIX., $\angle AGH = 68°$, how many degrees are there in BGH? in GHD? in DHF?

PROPOSITION XXIV. THEOREM.

79. *Two angles whose sides are parallel, each to each, are equal if both pairs of parallel sides extend in the same direction, or in opposite directions, from their vertices.*



Let AB be parallel to DH, and BC to KF.

I. To prove that the angles ABC and DEF, whose sides AB and DE, and also BC and EF, extend in the same direction from their vertices, are equal.

Let BC and DH intersect at G.

Then since the parallels AB and DE are cut by BC,

$$\angle ABC = \angle DGC.$$

[If two parallels are cut by a secant line, the corresponding angles are equal.] (§ 74.)

In like manner, $\angle DGC = \angle DEF$.

Whence, $\angle ABC = \angle DEF$. (1)

II. To prove that the angles ABC and HEK, whose sides AB and EH, and also BC and EK, extend in opposite directions from their vertices, are equal.

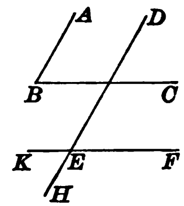
From (1), $\angle ABC = \angle DEF$.

But, $\angle DEF = \angle HEK$.

[If two straight lines intersect, the vertical angles are equal.] (§ 39.)

Whence, $\angle ABC = \angle HEK$.

80. Cor. *Two angles whose sides are parallel, each to each, are supplementary if one pair of parallel sides extends in the same direction, and the other pair in opposite directions, from their vertices.*



Let AB be parallel to DH, and BC to KF.

To prove that the angles ABC and DEK, whose sides AB and DE extend in the same direction, and BC and EK in opposite directions, from their vertices, are supplementary.

We have, $\angle ABC = \angle DEF$.

[Two angles whose sides are parallel, each to each, are equal if both pairs of parallel sides extend in the same direction from their vertices.] (§ 79.)

But $\angle DEF$ is the supplement of $\angle DEK$.

[If one straight line meet another, each of the adjacent angles formed is the supplement of the other.] (§ 35.)

Whence, $\angle ABC$ is the supplement of $\angle DEK$.

EXERCISES.

19. If, in the figure of Prop. XXIV., $\angle ABC = 59°$, how many degrees are there in each of the angles formed about the point E?

20. The straight line which bisects an angle bisects also its vertical angle.

21. If OD and OE are the bisectors of two complementary-adjacent angles, AOB and BOC, how many degrees are there in $\angle DOE$?

22. If from a point O in a straight line AB the lines OC and OD be drawn on opposite sides of AB, making $\angle AOC = \angle BOD$, prove that OC and OD lie in the same straight line. (§ 38.)

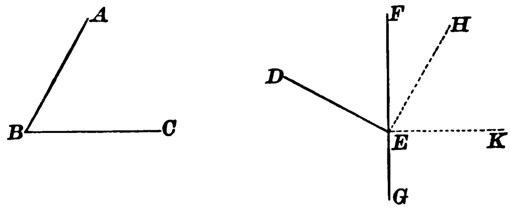
23. If, in a triangle ABC, $\angle A = \angle B$, a line parallel to AB makes equal angles with the sides AC and BC.

24. Two straight lines are parallel if any two points of either are equally distant from the other.

25. Any side of a triangle is less than the half-sum of the sides of the triangle. (§ 61.)

PROPOSITION XXV. THEOREM.

81. *Two angles whose sides are perpendicular, each to each, are either equal or supplementary.*



Let AB be perpendicular to DE, and BC to FG.

To prove that $\angle ABC$ is equal to $\angle DEF$, and supplementary to $\angle DEG$.

Draw EH and EK perpendicular to DE and EF, respectively.

Then EH and EK are parallel to AB and BC, respectively.

[Two perpendiculars to the same straight line are parallel.] (§ 54.)

Therefore, $\angle HEK = \angle ABC$.

[Two angles whose sides are parallel, each to each, are equal if both pairs of parallel sides extend in the same direction from their vertices.] (§ 79.)

But each of the angles HEK and DEF is the complement of FEH.

Whence, $\angle HEK = \angle DEF$.

[The complements of equal angles are equal.] (§ 33.)

Therefore, $\angle ABC = \angle DEF$.

Again, $\angle DEF$ is the supplement of $\angle DEG$.

[If one straight line meet another, each of the adjacent angles formed is the supplement of the other.] (§ 35.)

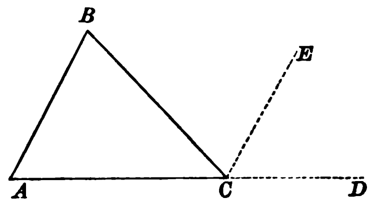
But, $\angle ABC = \angle DEF$.

Whence, $\angle ABC$ is the supplement of $\angle DEG$.

NOTE. The angles are equal if they are both acute or both obtuse; and supplementary if one is acute and the other obtuse.

Proposition XXVI. Theorem.

82. *The sum of the angles of any triangle is equal to two right angles.*



Let ABC be any triangle.

To prove $\angle A + \angle B + \angle C =$ two right angles.

Produce AC to D, and draw CE parallel to AB.

Then since the parallels AB and CE are cut by AD,

$$\angle ECD = \angle A.$$

[If two parallels are cut by a secant line, the corresponding angles are equal.] (§ 74.)

Again, $\angle BCE = \angle B$.

[If two parallels are cut by a secant line, the alternate-interior angles are equal.] (§ 72.)

But, $\angle ECD + \angle BCE + \angle ACB =$ two right angles.

[The sum of all the angles formed on the same side of a straight line at a given point is equal to two right angles.] (§ 36.)

Putting $\angle ECD = \angle A$, and $\angle BCE = \angle B$, we have

$$\angle A + \angle B + \angle ACB = \text{two right angles.}$$

83. Cor. I. It follows from the above demonstration that $\angle BCD = \angle ECD + \angle BCE = \angle A + \angle B$; hence,

1. *An exterior angle of a triangle is equal to the sum of the two opposite interior angles.*

2. *An exterior angle of a triangle is greater than either of the opposite interior angles.*

84. Cor. II. *A triangle cannot have two right angles, nor two obtuse angles.*

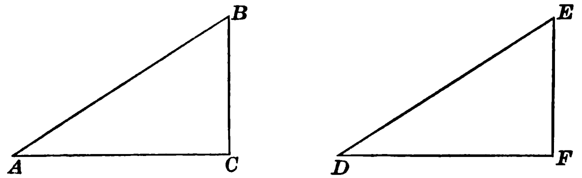
85. COR. III. *The sum of the acute angles of a right triangle is equal to one right angle.*

86. COR. IV. *If two triangles have two angles of one equal respectively to two angles of the other, the third angle of the first is equal to the third angle of the second.*

87. COR. V. *Two right triangles are equal when a leg and an acute angle of one are equal respectively to a leg and the homologous acute angle of the other;* for the remaining angles are equal by § 86, and then the theorem follows by § 68.

PROPOSITION XXVII. THEOREM.

88. *Two right triangles are equal when the hypotenuse and a leg of one are equal respectively to the hypotenuse and a leg of the other.*



In the right triangles ABC and DEF, let the hypotenuse AB be equal to DE, and the leg BC to EF.

To prove $\triangle ABC = \triangle DEF$.

Superpose ABC upon DEF in such a way that BC shall coincide with EF; the point B falling at E, and C at F.

Then since $\angle C = \angle F$, AC will fall upon DF.

But the equal oblique lines AB and DE cut off upon DF equal distances from the foot of the perpendicular EF.

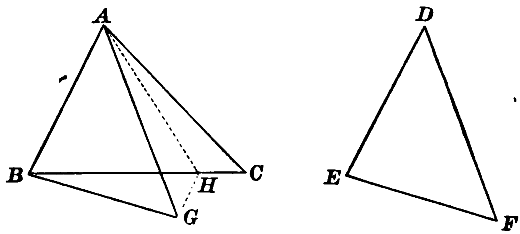
[If oblique lines be drawn from a point to a straight line, two equal oblique lines cut off equal distances from the foot of the perpendicular from the point to the line.] (§ 50.)

Whence, the point A falls at D.

Hence, ABC and DEF coincide throughout, and are equal.

PROPOSITION XXVIII. THEOREM.

89. *If two triangles have two sides of one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, the third side of the first is greater than the third side of the second.*



In the triangles ABC and DEF, let

$$AB = DE,\ AC = DF, \text{ and } \angle\ BAC > \angle\ EDF.$$

To prove $BC > EF.$

Place the triangle DEF in the position ABG; the side DE coinciding with its equal AB, and the vertex F falling at G.

Draw AH bisecting $\angle\ CAG$; also draw GH.

Then in the triangles ACH and AGH, AH is common.

Also, by hypothesis, $AC = AG$;

and by construction, $\angle\ CAH = \angle\ GAH.$

Therefore, $\triangle\ ACH = \triangle\ AGH.$

[Two triangles are equal when two sides and the included angle of one are equal respectively to two sides and the included angle of the other.] (§ 63.)

Whence, $CH = GH.$

[In equal figures, the homologous parts are equal.] (§ 66.)

But, $BH + GH > BG.$

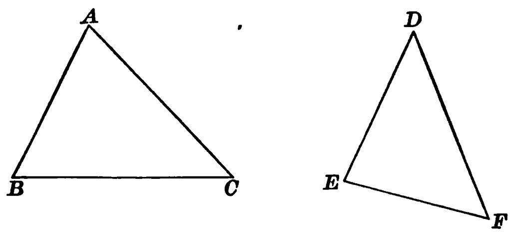
[A straight line is the shortest line between two points.] (Ax. 6.)

Substituting for GH its equal CH, we have

$$BH + CH > BG, \text{ or } BC > EF.$$

PROPOSITION XXIX. THEOREM.

90. (Converse of Prop. XXVIII.) *If two triangles have two sides of one equal respectively to two sides of the other, but the third side of the first greater than the third side of the second, the included angle of the first is greater than the included angle of the second.*



In the triangles ABC and DEF, let

$$AB = DE, AC = DF, \text{ and } BC > EF.$$

To prove $\angle A > \angle D$.

If $\angle A$ were equal to $\angle D$, the triangles ABC and DEF would be equal.

[Two triangles are equal when two sides and the included angle of one are equal respectively to two sides and the included angle of the other.] (§ 63.)

Then BC would be equal to EF.

[In equal figures, the homologous parts are equal.] (§ 66.)

Again, if $\angle A$ were less than $\angle D$, BC would be less than EF.

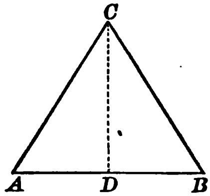
[If two triangles have two sides of one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, the third side of the first is greater than the third side of the second.] (§ 89.)

Each of these conclusions is contrary to the hypothesis that BC is greater than EF.

Therefore, $\angle A > \angle D$.

Proposition XXX. Theorem.

91. *In an isosceles triangle, the angles opposite the equal sides are equal.*



Let AC and BC be the equal sides of the isosceles triangle ABC.

To prove $\angle A = \angle B$.

Draw CD perpendicular to AB.

Then in the right triangles ACD and BCD, CD is common.

And by hypothesis, $AC = BC$.

Therefore, $\triangle ACD = \triangle BCD$.

[Two right triangles are equal when the hypotenuse and a leg of one are equal respectively to the hypotenuse and a leg of the other.] (§ 88.)

Whence, $\angle A = \angle B$.

[In equal figures, the homologous parts are equal.] (§ 66.)

92. Cor. I. From the equal triangles ACD and BCD we obtain,

$$AD = BD, \text{ and } \angle ACD = \angle BCD.$$

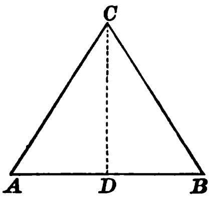
Hence, *the perpendicular from the vertex to the base of an isosceles triangle bisects the base, and also bisects the vertical angle.*

93. Cor. II. *An equilateral triangle is also equiangular.*

Ex. 26. The angles A and B of a triangle ABC are 57° and 98° respectively; how many degrees are there in the exterior angle at the vertex C?

PROPOSITION XXXI. THEOREM.

94. (Converse of Prop. XXX.) *If two angles of a triangle are equal, the sides opposite are equal.*



In the triangle ABC, let

$$\angle A = \angle B.$$

To prove $\quad AC = BC.$

Draw CD perpendicular to AB.

Then in the right triangles ACD and BCD, CD is common.

And by hypothesis, $\quad \angle A = \angle B.$

Therefore, $\quad \triangle ACD = \triangle BCD.$

[Two right triangles are equal when a leg and an acute angle of one are equal respectively to a leg and the homologous acute angle of the other.] (§ 87.)

Whence, $\quad AC = BC.$

[In equal figures, the homologous parts are equal.] (§ 66.)

95. COR. *An equiangular triangle is also equilateral.*

EXERCISES.

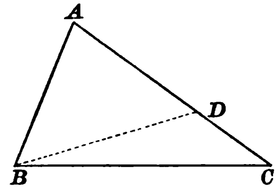
27. The angle B of a triangle ABC is three times the angle A, and the angle C is five times the angle A; how many degrees are there in each angle?

28. The exterior angles at the vertices A and B of a triangle ABC are 148° and 83° respectively; how many degrees are there in each angle of the triangle? How many degrees are there in the exterior angle at the vertex C?

29. How many degrees are there in each angle of an equiangular triangle?

PROPOSITION XXXII. THEOREM.

96. *In any triangle, the greater angle lies opposite the greater side.*



In the triangle ABC, let AC be greater than AB.

To prove $\angle ABC > \angle C$.

Lay off $AD = AB$, and draw BD.

Then, $\angle ABD = \angle ADB$.

[In an isosceles triangle, the angles opposite the equal sides are equal.] (§ 91.)

But, $\angle ADB > \angle C$.

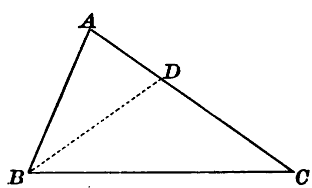
[An exterior angle of a triangle is greater than either of the opposite interior angles.] (§ 83.)

Whence, $\angle ABD > \angle C$.

Therefore, $\angle ABC > \angle C$.

PROPOSITION XXXIII. THEOREM.

97. (Converse of Prop. XXXII.) *In any triangle, the greater side lies opposite the greater angle.*



In the triangle ABC, let $\angle ABC$ be greater than $\angle C$.

To prove $AC > AB$.

Draw BD, making $\angle CBD = \angle C$.

Then, $BD = CD$.

[If two angles of a triangle are equal, the sides opposite are equal.] (§ 94.)

But, $AD + BD > AB$.

[A straight line is the shortest line between two points.] (Ax. 6.)

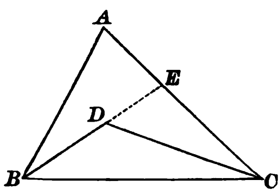
Substituting for BD its equal CD, we have

$$AD + CD > AB.$$

That is, $AC > AB$.

Proposition XXXIV. Theorem.

98. *If straight lines be drawn from a point within a triangle to the extremities of any side, the angle included by them is greater than the angle included by the other two sides.*



Let D be any point within the triangle ABC, and draw BD and CD.

To prove $\angle BDC > \angle A$.

Produce BD to meet AC at E.

Then, $\angle BDC > \angle DEC$.

[An exterior angle of a triangle is greater than either of the opposite interior angles.] (§ 83.)

In like manner, we have

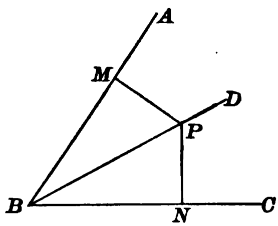
$$\angle DEC > \angle A.$$

Therefore, $\angle BDC > \angle A$.

Ex. 30. Prove Prop. XXX. by drawing CD so as to bisect $\angle ABC$.

PROPOSITION XXXV. THEOREM.

99. *Any point in the bisector of an angle is equally distant from the sides of the angle.*



Let P be any point in the bisector BD of the angle ABC, and draw PM and PN perpendicular to AB and BC, respectively.

To prove $PM = PN$.

In the right triangles BPM and BPN, BP is common.

And by hypothesis, $\angle PBM = \angle PBN$.

Therefore, $\triangle BPM = \triangle BPN$.

[Two right triangles are equal when the hypotenuse and an adjacent angle of one are equal respectively to the hypotenuse and an adjacent angle of the other.] (§ 70.)

Whence, $PM = PN$.

[In equal figures, the homologous parts are equal.] (§ 66.)

EXERCISES.

31. The angle at the vertex of an isosceles triangle ABC is equal to five-thirds the sum of the equal angles B and C. How many degrees are there in each angle ?

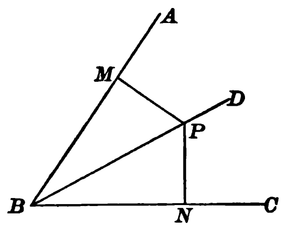
32. If the equal sides of an isosceles triangle be produced, the exterior angles formed with the base are equal.

33. The bisector of the vertical angle of an isosceles triangle bisects the base at right angles.

34. The line joining the vertex of an isosceles triangle to the middle point of the base, is perpendicular to the base, and bisects the vertical angle.

Proposition XXXVI. Theorem.

100. (Converse of Prop. XXXV.) *Every point which is within an angle, and equally distant from its sides, lies in the bisector of the angle.*



Let the point P be within the angle ABC, and equally distant from its sides; and draw BP.

To prove that BP bisects $\angle ABC$.

Draw PM and PN perpendicular to AB and BC, respectively.

Then in the right triangles BPM and BPN, BP is common.

And by hypothesis, $PM = PN$.

Therefore, $\triangle BPM = \triangle BPN$.

[Two right triangles are equal when the hypotenuse and a leg of one are equal respectively to the hypotenuse and a leg of the other.] (§ 88.)

Whence, $\angle PBM = \angle PBN$.

[In equal figures, the homologous parts are equal.] (§ 66.)

Therefore, BP bisects $\angle ABC$.

EXERCISES.

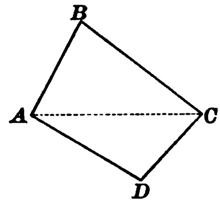
35. The exterior angle at the vertex of an isosceles triangle is 101°; how many degrees are there in the exterior angles at the extremities of the base?

36. If the perpendicular from the vertex to the base of a triangle bisects the base, the triangle is isosceles.

QUADRILATERALS.

Definitions.

101. A *quadrilateral* is a portion of a plane bounded by four straight lines; as *ABCD*.



The bounding lines are called the *sides* of the quadrilateral, and their points of intersection the *vertices*.

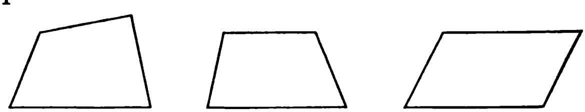
The *angles* of the quadrilateral are the angles formed by the adjacent sides.

A *diagonal* is a straight line joining two opposite vertices; as *AC*.

102. A *Trapezium* is a quadrilateral no two of whose sides are parallel.

A *Trapezoid* is a quadrilateral two, and only two, of whose sides are parallel.

A *Parallelogram* is a quadrilateral whose opposite sides are parallel.



Trapezium. *Trapezoid.* *Parallelogram.*

The *bases* of a trapezoid are its parallel sides; the *altitude* is the perpendicular distance between them.

The *bases* of a parallelogram are the side on which it is supposed to stand and the parallel side; the *altitude* is the perpendicular distance between them.

103. A *Rhomboid* is a parallelogram whose angles are not right angles, and whose adjacent sides are unequal.

A *Rhombus* is a parallelogram whose angles are not right angles, and whose adjacent sides are equal.

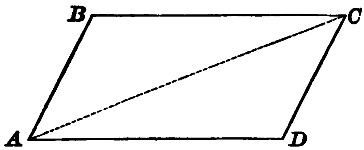
A *Rectangle* is a parallelogram whose angles are right angles.

A *Square* is a rectangle whose sides are equal.

Rhomboid. *Rhombus.* *Rectangle.* *Square.*

Proposition XXXVII. Theorem.

104. *The opposite sides of a parallelogram are equal.*



Let $ABCD$ be a parallelogram.

To prove $AB = CD$, and $BC = AD$.

Draw the diagonal AC.

Then in the triangles ABC and ACD, AC is common.

Again, since the parallels BC and AD are cut by AC,

$$\angle BCA = \angle CAD.$$

[If two parallels are cut by a secant line, the alternate-interior angles are equal.] (§ 72.)

And since the parallels AB and CD are cut by AC,

$$\angle BAC = \angle ACD.$$

Therefore, $\triangle ABC = \triangle ACD$.

[Two triangles are equal when a side and two adjacent angles of one are equal respectively to a side and two adjacent angles of the other.] (§ 68.)

Whence, $AB = CD$, and $BC = AD$.

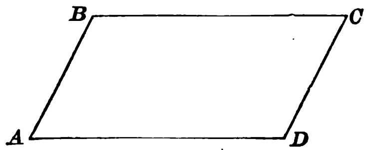
[In equal figures, the homologous parts are equal.] (§ 66.)

105. Cor. I. *Parallel lines included between parallel lines are equal.*

106. Cor. II. *A diagonal of a parallelogram divides it into two equal triangles.*

PROPOSITION XXXVIII. THEOREM.

107. *The opposite angles of a parallelogram are equal.*



Let $ABCD$ be a parallelogram.

To prove $\angle A = \angle C$, and $\angle B = \angle D$.

Since AB is parallel to CD, and AD to BC, we have

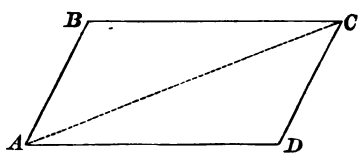
$$\angle A = \angle C.$$

[Two angles whose sides are parallel, each to each, are equal if both pairs of parallel sides extend in opposite directions from their vertices.] (§ 79.)

In like manner, we may prove $\angle B = \angle D$.

PROPOSITION XXXIX. THEOREM.

108. (Converse of Prop. XXXVII.) *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*



In the quadrilateral $ABCD$, let $AB = CD$ and $BC = AD$.

To prove $ABCD$ a parallelogram.

Draw AC.

Then in the triangles ABC and ACD, AC is common.

And by hypothesis, $AB = CD$, and $BC = AD$.

Therefore, $\triangle ABC = \triangle ACD$.

[Two triangles are equal when the three sides of one are equal respectively to the three sides of the other.] (§ 69.)

Whence, $\angle BCA = \angle CAD$, and $\angle BAC = \angle ACD$.

[In equal figures, the homologous parts are equal.] (§ 66.)

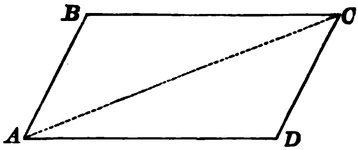
Then BC is parallel to AD, and AB to CD.

[If two straight lines are cut by a secant line, making the alternate-interior angles equal, the two lines are parallel.] (§ 73.)

Therefore, $ABCD$ is a parallelogram.

PROPOSITION XL. THEOREM.

109. *If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*



In the quadrilateral $ABCD$, let BC be equal and parallel to AD.

To prove $ABCD$ a parallelogram.

Draw AC.

Then in the triangles ABC and ACD, AC is common.

And by hypothesis, $BC = AD$.

Also, since the parallels BC and AD are cut by AC,

$$\angle BCA = \angle CAD.$$

[If two parallels are cut by a secant line, the alternate-interior angles are equal.] (§ 72.)

Therefore, $\triangle ABC = \triangle ACD$.

[Two triangles are equal when two sides and the included angle of one are equal respectively to two sides and the included angle of the other.] (§ 63.)

Whence, $AB = CD$.

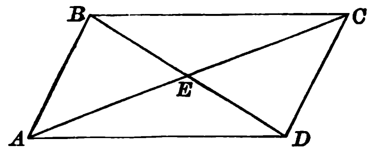
[In equal figures, the homologous parts are equal.] (§ 66.)

Therefore, $ABCD$ is a parallelogram.

[If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.] (§ 108.)

Proposition XLI. Theorem.

110. *The diagonals of a parallelogram bisect each other.*



Let the diagonals AC and BD of the parallelogram $ABCD$ intersect at E.

To prove $AE = EC$, and $BE = ED$.

In the triangles AED and BEC, $AD = BC$.

[The opposite sides of a parallelogram are equal.] (§ 104.)

And since the parallels AD and BC are cut by AC,

$$\angle EAD = \angle ECB.$$

[If two parallels are cut by a secant line, the alternate-interior angles are equal.] (§ 72.)

In like manner, $\angle EDA = \angle EBC$.

Therefore, $\triangle AED = \triangle BEC$.

[Two triangles are equal when a side and two adjacent angles of one are equal respectively to a side and two adjacent angles of the other.] (§ 68.)

Whence, $AE = EC$, and $BE = ED$.

[In equal figures, the homologous parts are equal.] (§ 66.)

Note. The point E is called the *centre* of the parallelogram.

EXERCISES.

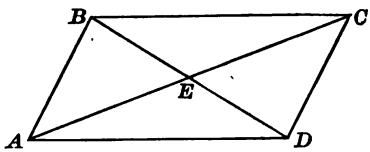
37. If one angle of a parallelogram is 119°, how many degrees are there in each of the others?

38. If one angle of a parallelogram is a right angle, the figure is a rectangle.

39. If from any point in the base of an isosceles triangle perpendiculars to the equal sides be drawn, they make equal angles with the base.

PROPOSITION XLII. THEOREM.

111. (Converse of Prop. XLI.) *If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.*



Let the diagonals AC and BD of the quadrilateral $ABCD$ bisect each other at E.

To prove $ABCD$ a parallelogram.

In the triangles AED and BEC, by hypothesis,

$$AE = EC, \text{ and } DE = EB.$$

Also, $\angle AED = \angle BEC.$

[If two straight lines intersect, the vertical angles are equal.] (§ 39.)

Therefore, $\triangle AED = \triangle BEC.$

[Two triangles are equal when two sides and the included angle of one are equal respectively to two sides and the included angle of the other.] (§ 63.)

Whence, $AD = BC.$

[In equal figures, the homologous parts are equal.] (§ 66.)

In like manner, $\triangle AEB = \triangle CED,$

and $AB = CD.$

Therefore, $ABCD$ is a parallelogram.

[If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.] (§ 108.)

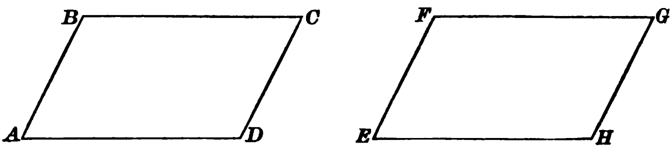
EXERCISES.

40. If the angles adjacent to one base of a trapezoid are equal, those adjacent to the other base are also equal.

41. If two parallels are cut by a secant line, the bisectors of the four interior angles form a rectangle.

PROPOSITION XLIII. THEOREM.

112. *Two parallelograms are equal when two adjacent sides and the included angle of one are equal respectively to two adjacent sides and the included angle of the other.*



In the parallelograms $ABCD$ and $EFGH$, let

$$AB = EF,\ AD = EH, \text{ and } \angle A = \angle E.$$

To prove

$$\text{parallelogram } ABCD = \text{parallelogram } EFGH.$$

Superpose the parallelogram $ABCD$ upon $EFGH$ in such a way that $\angle A$ shall coincide with its equal $\angle E$; the side AB falling upon EF, and the side AD upon EH.

Then since $AB = EF$ and $AD = EH$, the point B will fall at F, and the point D at H.

Now since BC is parallel to AD, and FG to EH, the side BC will fall upon FG, and the point C will fall somewhere in FG.

[But one straight line can be drawn through a given point parallel to a given straight line.] (§ 53.)

In like manner, the side DC will fall upon HG, and the point C will fall somewhere in HG.

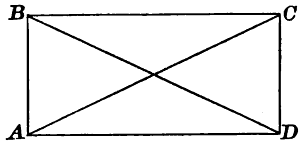
Therefore the point C, falling at the same time in FG and HG, must fall at their intersection, G.

Hence, $ABCD$ and $EFGH$ coincide throughout, and are equal.

113. COR. *Two rectangles are equal when the base and altitude of one are equal respectively to the base and altitude of the other.*

Proposition XLIV. Theorem.

114. *The diagonals of a rectangle are equal.*



Let $ABCD$ be a rectangle.

To prove diagonal $AC =$ diagonal BD.

In the right triangles ABD and ACD, AD is common.

Also, $AB = CD$.

[The opposite sides of a parallelogram are equal.] (§ 104.)

Therefore, $\triangle ABD = \triangle ACD$.

[Two triangles are equal when two sides and the included angle of one are equal respectively to two sides and the included angle of the other.] (§ 63.)

Whence, $AC = BD$.

[In equal figures, the homologous parts are equal.] (§ 66.)

115. Cor. *The diagonals of a square are equal.*

EXERCISES.

42. If the diagonals of a parallelogram are equal, the figure is a rectangle.

43. If two adjacent sides of a quadrilateral are equal, and the diagonal bisects their included angle, the other two sides are equal.

44. The bisectors of the interior angles of a parallelogram form a rectangle.

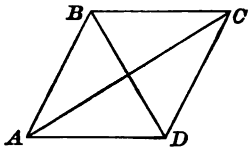
45. The bisectors of the interior angles of a trapezoid form a quadrilateral, two of whose angles are right angles.

46. If the angles at the base of a trapezoid are equal, the non-parallel sides are also equal.

47. If the non-parallel sides of a trapezoid are equal, the angles which they make with the bases are equal.

PROPOSITION XLV. THEOREM.

116. *The diagonals of a rhombus bisect each other at right angles.*



Let $ABCD$ be a rhombus.

To prove that its diagonals AC and BD bisect each other at right angles.

Since the adjacent sides of a rhombus are equal,

$$AB = AD, \text{ and } BC = CD.$$

Whence, AC is perpendicular to BD at its middle point.

[Two points, each equally distant from the extremities of a straight line, determine a perpendicular at its middle point.] (§ 43.)

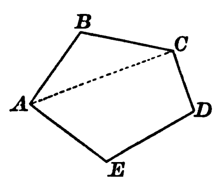
In like manner, it may be proved that BD is perpendicular to AC at its middle point.

Hence, AC and BD bisect each other at right angles.

POLYGONS.

DEFINITIONS.

117. A *polygon* is a portion of a plane bounded by three or more straight lines; as $ABCDE$.



The bounding lines are called the *sides* of the polygon, and their sum is called the *perimeter*.

The *angles* of the polygon are the angles EAB, ABC, etc., formed by the adjacent sides; and their vertices are called the *vertices* of the polygon.

A *diagonal* of a polygon is a straight line joining any two vertices which are not consecutive; as AC.

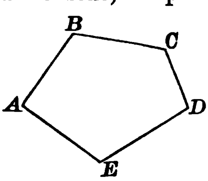
118. Polygons are classified with reference to the number of their sides, as follows:

No. of Sides.	Designation.	No. of Sides.	Designation.
3	Triangle.	8	Octagon.
4	Quadrilateral.	9	Enneagon.
5	Pentagon.	10	Decagon.
6	Hexagon.	11	Undecagon.
7	Heptagon.	12	Dodecagon.

119. An *equilateral* polygon is a polygon all of whose sides are equal.

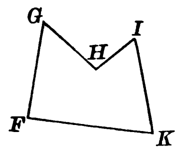
An *equiangular* polygon is a polygon all of whose angles are equal.

120. A polygon is called *convex* when no side, if produced, will enter the space enclosed by the perimeter; as $ABCDE$.



It is evident that, in such a case, each angle of the polygon is less than two right angles.

A polygon is called *concave* when at least two of its sides, if produced, will enter the space enclosed by the perimeter; as $FGHIK$.



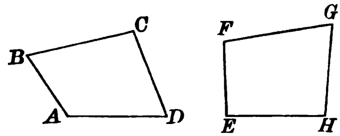
It is evident that, in such a case, at least one angle of the polygon is greater than two right angles.

Thus in the polygon $FGHIK$, the interior angle whose vertex is H is greater than two right angles.

Such an angle is called *re-entrant*.

All polygons considered hereafter will be understood to be convex, unless the contrary is stated.

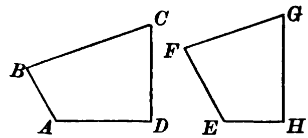
121. Two polygons are said to be *mutually equilateral* when the sides of one are equal respectively to the sides of the other, when taken in the same order.



Thus the polygons $ABCD$ and $EFGH$ are mutually equilateral if

$$AB = EF,\ BC = FG,\ CD = GH, \text{ and } DA = HE.$$

122. Two polygons are said to be *mutually equiangular* when the angles of one are equal respectively to the angles of the other, when taken in the same order.



Thus the polygons $ABCD$ and $EFGH$ are mutually equiangular if

$$\angle A = \angle E,\ \angle B = \angle F,\ \angle C = \angle G, \text{ and } \angle D = \angle H.$$

123. In polygons which are mutually equilateral or mutually equiangular, sides or angles which are similarly placed are called *homologous.*

If two *triangles* are mutually equilateral, they are also mutually equiangular (§ 69); but with this exception, two polygons may be mutually equilateral without being mutually equiangular, or mutually equiangular without being mutually equilateral.

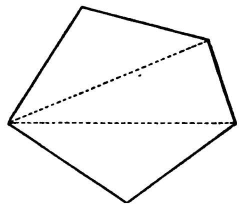
124. *If two polygons are both mutually equilateral and mutually equiangular, they are equal.*

For they can evidently be applied one to the other so as to coincide throughout.

125. Two polygons are equal when they are composed of the same number of triangles, equal each to each, and similarly placed; for they can evidently be applied one to the other so as to coincide throughout.

Proposition XLVI. Theorem.

126. *The sum of the angles of any polygon is equal to two right angles taken as many times, less two, as the polygon has sides.*



Any polygon may be divided into triangles by drawing diagonals from one of its vertices; the number of triangles being equal to the number of sides of the polygon, less two.

The sum of the angles of the polygon is equal to the sum of the angles of the triangles.

And the sum of the angles of each triangle is equal to two right angles.

[The sum of the angles of any triangle is equal to two right angles.] (§ 82.)

Hence, the sum of the angles of the polygon is equal to two right angles taken as many times, less two, as the polygon has sides.

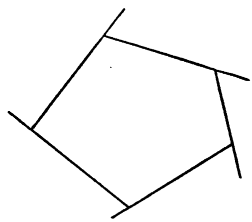
127. Cor. I. *The sum of the angles of any polygon is equal to twice as many right angles as the polygon has sides, less four right angles.*

Thus, if R represents a right angle, and n the number of sides of a polygon, the sum of its angles is expressed by $2\,nR - 4\,R$.

128. Cor. II. *The sum of the angles of a quadrilateral is equal to four right angles; of a pentagon, six right angles; of a hexagon, eight right angles; etc.*

Proposition XLVII. Theorem.

129. *If the sides of any polygon be produced so as to form an exterior angle at each vertex, the sum of these exterior angles is equal to four right angles.*



The sum of the exterior and interior angles formed at any one vertex is equal to two right angles.

[If one straight line meet another, the sum of the adjacent angles formed is equal to two right angles.] (§ 34.)

Hence, the sum of *all* the exterior and interior angles is equal to twice as many right angles as the polygon has sides.

But the sum of the interior angles alone is equal to twice as many right angles as the polygon has sides, less four right angles.

[The sum of the angles of any polygon is equal to twice as many right angles as the polygon has sides, less four right angles.] (§ 127.)

Whence, the sum of the exterior angles is equal to four right angles.

EXERCISES.

48. How many degrees are there in each angle of an equiangular hexagon? of an equiangular octagon? of an equiangular decagon? of an equiangular dodecagon?

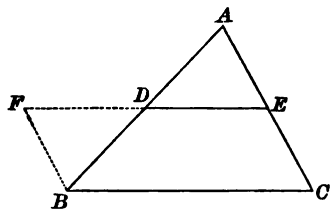
49. How many degrees are there in the exterior angle at each vertex of an equiangular pentagon?

50. If two angles of a quadrilateral are supplementary, the other two angles are supplementary.

MISCELLANEOUS THEOREMS.

PROPOSITION XLVIII. THEOREM.

130. *The line joining the middle points of two sides of a triangle is parallel to the third side, and equal to one-half of it.*



Let DE bisect the sides AB and AC of the triangle ABC.

To prove DE parallel to BC, and equal to $\frac{1}{2} BC$.

Draw BF parallel to AC, meeting ED produced at F.

Then in the triangles ADE and BDF, $\angle A = \angle DBF$.

[If two parallels are cut by a secant line, the alternate-interior angles are equal.] (§ 72.)

Also, $\angle ADE = \angle BDF$.

[If two straight lines intersect, the vertical angles are equal.] (§ 39.)

And by hypothesis, $AD = BD$.

Therefore, $\triangle ADE = \triangle BDF$.

[Two triangles are equal when a side and two adjacent angles of one are equal respectively to a side and two adjacent angles of the other.] (§ 68.)

Whence, $DE = DF$, and $AE = BF$.

[In equal figures, the homologous parts are equal.] (§ 66.)

Then, since $AE = EC$, BF is equal and parallel to CE.

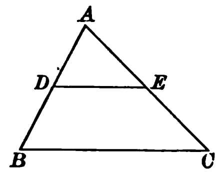
Whence, $BCEF$ is a parallelogram.

[If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.] (§ 109.)

Hence, DE is parallel to BC, and $DE = \frac{1}{2} FE = \frac{1}{2} BC$.

[The opposite sides of a parallelogram are equal.] (§ 104.)

131. Cor. *The line which bisects one side of a triangle, and is parallel to another side, bisects also the third side.*



In the triangle ABC, let DE be parallel to BC, and bisect AB.

To prove that DE bisects AC.

A line drawn from D to the middle point of AC will be parallel to BC.

[The line joining the middle points of two sides of a triangle is parallel to the third side.] (§ 130.)

Then this line will coincide with DE.

[But one straight line can be drawn through a given point parallel to a given straight line.] (§ 53.)

Therefore, DE bisects AC.

EXERCISES.

51. The bisectors of the equal angles of an isosceles triangle form, with the base, another isosceles triangle.

52. Either exterior angle at the base of an isosceles triangle is equal to the sum of a right angle and one-half the vertical angle.

53. The straight lines bisecting the equal angles of an isosceles triangle, and terminating in the opposite sides, are equal.

54. The perpendiculars from the extremities of the base of an isosceles triangle to the opposite sides are equal.

55. The angle between the bisectors of the equal angles of an isosceles triangle is equal to the exterior angle at the base of the triangle.

56. If through a point midway between two parallels two secant lines be drawn, they intercept equal portions of the parallels.

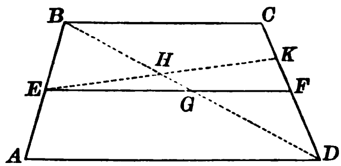
57. If a line joining two parallels be bisected, any line drawn through the point of bisection and included between the parallels will be bisected at the point.

58. If perpendiculars BE and DF be drawn from the vertices B and D of the parallelogram $ABCD$ to the diagonal AC, prove that $BE = DF$.

59. The lines joining the middle points of the sides of a triangle divide it into four equal triangles. (§ 130.)

PROPOSITION XLIX. THEOREM.

132. *The line joining the middle points of the non-parallel sides of a trapezoid is parallel to the bases, and equal to one-half their sum.*



Let the line EF bisect the non-parallel sides AB and CD of the trapezoid $ABCD$.

I. To prove EF parallel to AD and BC.

Let EK be parallel to AD and BC; and draw BD, intersecting EF at G, and EK at H.

Then in the triangle ABD, EH is parallel to AD and bisects AB; it therefore bisects BD.

[The line which bisects one side of a triangle, and is parallel to another side, bisects also the third side.] (§ 131.)

In like manner, in the triangle BCD, HK is parallel to BC and bisects BD; it therefore bisects CD.

But this is impossible unless EK coincides with EF.

Hence, EF is parallel to AD and BC.

II. To prove $EF = \frac{1}{2}(AD + BC)$.

Since EG bisects AB and BD,

$$EG = \tfrac{1}{2} AD. \qquad (1)$$

[The line joining the middle points of two sides of a triangle is equal to one-half the third side.] (§ 130.)

And since GF bisects BD and CD,

$$GF = \tfrac{1}{2} BC. \qquad (2)$$

Adding (1) and (2), we have,

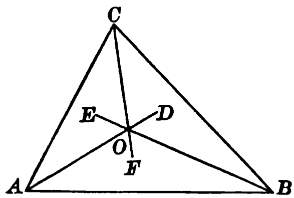
$$EG + GF = \tfrac{1}{2} AD + \tfrac{1}{2} BC.$$

That is, $EF = \frac{1}{2}(AD + BC)$.

133. COR. *The line which is parallel to the bases of a trapezoid, and bisects one of the non-parallel sides, bisects the other also.*

PROPOSITION L. THEOREM.

134. *The bisectors of the angles of a triangle meet in a common point.*



Let AD, BE, and CF be the bisectors of the angles A, B, and C of the triangle ABC.

To prove that AD, BE, and CF intersect in a common point.

Let AD and BE intersect at O.

Then since O is in the bisector AD, it is equally distant from the sides AB and AC.

[Any point in the bisector of an angle is equally distant from the sides of the angle.] (§ 99.)

In like manner, since O is in the bisector BE, it is equally distant from the sides AB and BC.

Then O is equally distant from the sides AC and BC, and therefore lies in the bisector CF.

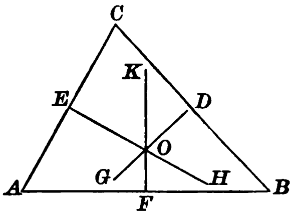
[Every point which is within an angle, and equally distant from its sides, lies in the bisector of the angle.] (§ 100.)

Hence, AD, BE, and CF meet in the common point O.

135. COR. *The point of intersection of the bisectors of the angles of a triangle is equally distant from the sides of the triangle.*

PROPOSITION LI. THEOREM.

136. *The perpendiculars erected at the middle points of the sides of a triangle meet in a common point.*



Let DG, EH, and FK be the perpendiculars erected at the middle points of the sides BC, CA, and AB, of the triangle ABC.

To prove that DG, EH, and FK intersect in a common point.

Let DG and EH intersect at O.

Then since O is in the perpendicular DG, it is equally distant from B and C.

[If a perpendicular be erected at the middle point of a straight line, any point in the perpendicular is equally distant from the extremities of the line.] (§ 40.)

In like manner, since O is in the perpendicular EH, it is equally distant from A and C.

Then O is equally distant from A and B, and therefore lies in the perpendicular FK.

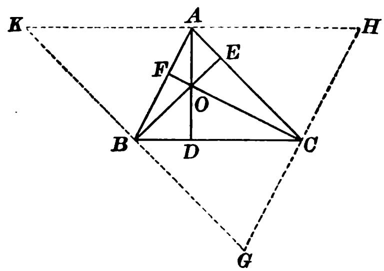
[Every point which is equally distant from the extremities of a straight line, lies in the perpendicular erected at the middle point of the line.] (§ 42.)

Therefore, DG, EH, and FK meet in the common point O.

137. COR. *The point of intersection of the perpendiculars erected at the middle points of the sides of a triangle, is equally distant from the vertices of the triangle.*

PROPOSITION LII. THEOREM.

138. *The perpendiculars from the vertices of a triangle to the opposite sides meet in a common point.*



Let AD, BE, and CF be the perpendiculars from the vertices of the triangle ABC to the opposite sides.

To prove that AD, BE, and CF meet in a common point.

Through A, B, and C, draw HK, KG, and GH, parallel, respectively, to BC, CA, and AB.

Then since AD is perpendicular to BC, it is also perpendicular to HK.

[A straight line perpendicular to one of two parallels is perpendicular to the other.] (§ 56.)

Now since $ABCH$ and $ACBK$ are parallelograms,

$$AH = BC, \text{ and } AK = BC.$$

[The opposite sides of a parallelogram are equal.] (§ 104.)

Whence, $AH = AK$, and A is the middle point of HK.

Then AD is perpendicular to HK at its middle point.

In like manner, BE and CF are the perpendiculars erected at the middle points of KG and GH, respectively.

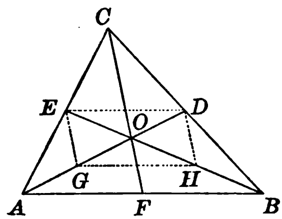
Hence, AD, BE, and CF meet in a common point O.

[The perpendiculars erected at the middle points of the sides of a triangle meet in a common point.] (§ 136.)

139. DEF. A *median* of a triangle is a line drawn from any vertex to the middle point of the opposite side.

PROPOSITION LIII. THEOREM.

140. *The medians of a triangle meet in a common point.*



Let AD, BE, and CF be the medians of the triangle ABC.

To prove that AD, BE, and CF meet in a common point.

Let AD and BE intersect at O.

Let G and H be the middle points of OA and OB, respectively, and draw ED, GH, EG, and DH.

Then since ED bisects AC and BC, it is parallel to AB, and equal to $\frac{1}{2}$ AB.

[The line joining the middle points of two sides of a triangle is parallel to the third side, and equal to one-half of it.] (§ 130.)

In like manner, since GH bisects OA and OB, it is parallel to AB, and equal to $\frac{1}{2}$ AB.

Therefore, ED and GH are equal and parallel, and $EDHG$ is a parallelogram.

[If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.] (§ 109.)

Then GD and EH bisect each other at O.

[The diagonals of a parallelogram bisect each other.] (§ 110.)

But by hypothesis, G is the middle point of OA.

Hence, $AG = OG = OD$, and $OA = \frac{2}{3}$ AD.

That is, BE intersects AD two-thirds the way from A to BC.

In like manner, it may be proved that CF intersects AD two-thirds the way from A to BC.

Hence, AD, BE, and CF meet in the common point O.

LOCI.

141. DEF. If a series of points, all of which satisfy a given condition, lie in a certain line, that line is called the *locus* of the points.

For example, every point which satisfies the condition of being equally distant from the extremities of a straight line, lies in the perpendicular erected at the middle point of the line (§ 42).

Hence, *the perpendicular erected at the middle point of a straight line is the* LOCUS *of points which are equally distant from the extremities of the line.*

Again, every point which satisfies the condition of being within an angle, and equally distant from its sides, lies in the bisector of the angle (§ 100).

Hence, *the bisector of an angle is the locus of points which are within the angle, and equally distant from its sides.*

EXERCISES.

60. What is the locus of points at a given distance from a given straight line?

61. What is the locus of points equally distant from a pair of intersecting straight lines?

62. What is the locus of points equally distant from a pair of parallel straight lines?

63. Two isosceles triangles are equal when the base and vertical angle of one are equal respectively to the base and vertical angle of the other.

64. If from any point in the base of an isosceles triangle parallels to the equal sides be drawn, the perimeter of the parallelogram formed is equal to the sum of the equal sides of the triangle.

65. If an exterior angle be formed at the vertex of an isosceles triangle, its bisector is parallel to the base.

66. The medians drawn from the extremities of the base of an isosceles triangle are equal.

67. State and prove the converse of Ex. 54, p. 60.

68. The diagonals of a rhombus bisect its angles.

69. If from the vertex of one of the equal angles of an isosceles triangle a perpendicular be drawn to the opposite side, it makes with the base an angle which is equal to one-half the vertical angle of the triangle.

70. Every point within an angle, and not in the bisector, is unequally distant from the sides of the angle.

71. If from any point in the bisector of an angle a parallel to one of the sides be drawn, the bisector, the parallel, and the remaining side form an isosceles triangle.

72. If the bisectors of the equal angles of an isosceles triangle meet the equal sides at D and E, prove that DE is parallel to the base of the triangle.

73. If the exterior angles at the vertices A and B of a triangle ABC are bisected by lines which meet at D, prove that

$$\angle ADB = 90^\circ - \tfrac{1}{2} C.$$

74. If at any point D in one of the equal sides AB of the isosceles triangle ABC, DE is drawn perpendicular to the base BC meeting CA produced at E, prove that the triangle ADE is isosceles.

75. From C, one of the extremities of the base BC of an isosceles triangle ABC, a line is drawn meeting BA produced at D, making $AD = AB$. Prove that CD is perpendicular to BC. (§ 91.)

76. If the non-parallel sides of a trapezoid are equal, its diagonals are also equal.

77. If ADC is a re-entrant angle of the quadrilateral $ABCD$, prove that the angle ADC, exterior to the figure, is equal to the sum of the interior angles A, B, and C.

78. If a diagonal of a quadrilateral bisects two of its angles, it is perpendicular to the other diagonal.

79. If two lines are cut by a third so as to make the sum of the interior angles on the same side of the secant line less than two right angles, the lines will meet if sufficiently produced.

80. If exterior angles be formed at two vertices of a triangle, their bisectors will intersect on the bisector of the interior angle at the third vertex. (§ 134.)

81. In a quadrilateral $ABCD$, the angles ABD and CAD are equal to ACD and BDA respectively; prove that the figure is a trapezoid.

82. $ABCD$ is a trapezoid whose parallel sides AD and BC are perpendicular to CD. If E is the middle point of AB, prove that $EC = ED$.

83. State and prove the converse of Prop. XLV.

84. State and prove the converse of Ex. 65, p. 66.

85. Prove Prop. XXVI. by drawing through B a parallel to AC.

86. Prove Prop. XXX. by drawing CD to the middle point of AB.

87. Prove Prop. XXXI. by drawing CD so as to bisect $\angle ACB$.

88. The middle point of the hypotenuse of a right triangle is equally distant from the vertices of the triangle.

89. The bisectors of the angles of a rectangle form a square.

90. If D is the middle point of the side BC of the triangle ABC, and BE and CF are the perpendiculars from B and C to AD, produced if necessary, prove that $BE = CF$.

91. The angle at the vertex of an isosceles triangle ABC is equal to twice the sum of the equal angles B and C. If CD be drawn perpendicular to BC, meeting BA produced at D, prove that the triangle ACD is equilateral.

92. The bisector of the vertical angle A of an equilateral triangle ABC is produced to D, so that $AD = AB$. If BD and CD be drawn, prove that $\angle BDC$ is 30° or 150°, according as D lies above or below the base.

93. If the angle B of the triangle ABC is greater than the angle C, and BD be drawn to AC making $AD = AB$, prove that

$$\angle ADB = \tfrac{1}{2}(B + C), \text{ and } \angle CBD = \tfrac{1}{2}(B - C).$$

94. The sum of the lines drawn from any point within a triangle to the vertices is greater than the half-sum of the three sides. (§ 61.)

95. The sum of the lines drawn from any point within a triangle to the vertices is less than the sum of the three sides.

96. How many sides are there in the polygon the sum of whose interior angles exceeds the sum of its exterior angles by 540°?

97. If D, E, and F are points on the sides AB, BC, and CA of an equilateral triangle ABC, such that $AD = BE = CF$, prove that the figure DEF is an equilateral triangle.

98. If E, F, G, and H are points on the sides AB, BC, CD, and DA of a parallelogram $ABCD$, such that $AE = CG$ and $BF = DH$, prove that the figure $EFGH$ is a parallelogram.

99. If E, F, G, and H are points on the sides AB, BC, CD, and DA of a square $ABCD$, such that $AE = BF = CG = DH$, prove that the figure $EFGH$ is a square.

100. If on the diagonal BD of a square $ABCD$ a distance BE is taken equal to AB, and EF is drawn perpendicular to BD, meeting AD at F, prove that $AF = EF = ED$.

101. Prove the theorem of § 127 by drawing lines from any point within the polygon to the vertices.

102. State and prove the converse of Ex. 68, p. 66.

103. State and prove the converse of Prop. XXXVIII.

104. State and prove the converse of Ex. 76, p. 67.

105. If AD is the perpendicular from the vertex of the right angle to the hypotenuse of the right triangle ABC, and AE is the bisector of the angle A, prove that $\angle DAE$ is equal to one-half the difference of the angles B and C.

106. D is any point in the base BC of an isosceles triangle ABC. The side AC is produced below C to E, so that $CE = CD$, and DE is drawn meeting AB at F. Prove that $\angle AFE = 3 \angle AEF$.

107. If ABC and ABD are two triangles on the same base and on the same side of it, such that $AC = BD$ and $AD = BC$, and AD and BC intersect at O, prove that the triangle OAB is isosceles.

108. If D is the middle point of the side AC of the equilateral triangle ABC, and DE be drawn perpendicular to BC, prove that $EC = \frac{1}{4} BC$.

109. If in the parallelogram $ABCD$, E and F are the middle points of the sides BC and AD, prove that the lines AE and CF trisect the the diagonal BD.

110. If AD is the perpendicular from the vertex of the right angle to the hypotenuse of the right triangle ABC, and E is the middle point of BC, prove that $\angle DAE$ is equal to the difference of the angles B and C.

111. If one acute angle of a right triangle is double the other, the hypotenuse is double the shortest leg.

112. If AD be drawn from the vertex of the right angle to the hypotenuse of the right triangle ABC so as to make $\angle DAC = \angle C$, it bisects the hypotenuse.

113. If D is the middle point of the side BC of the triangle ABC, prove that $AD > \frac{1}{2}(AB + AC - BC)$. (§ 62.)

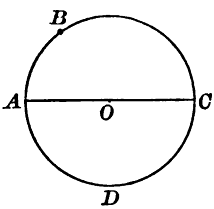
NOTE. For additional exercises on Book I., see p. 221.

BOOK II.

THE CIRCLE.

DEFINITIONS.

142. A *circle* is a portion of a plane bounded by a curve called a *circumference*, all points of which are equally distant from a point within, called the *centre;* as $ABCD$.



Any portion of the circumference, as AB, is called an *arc*.

A *radius* is a straight line drawn from the centre to the circumference; as OA.

A *diameter* is a straight line drawn through the centre, having its extremities in the circumference; as AC.

143. It follows from the definition of § 142 that,

All radii of a circle are equal.

Also, all its diameters are equal, since each is the sum of two radii.

144. *Two circles are equal when their radii are equal.*

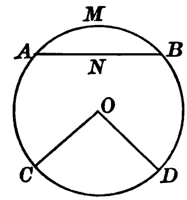
For they can evidently be applied one to the other so that their circumferences shall coincide throughout.

145. Conversely, *the radii of equal circles are equal.*

146. A *semi-circumference* is an arc equal to one-half the circumference; and a *quadrant* is an arc equal to one-fourth the circumference.

Concentric circles are circles having the same centre.

147. A *chord* is a straight line joining the extremities of an arc; as *AB*.



The arc is said to be *subtended* by its chord.

Every chord subtends two arcs; thus the chord *AB* subtends the arcs *AMB* and and *ACDB*.

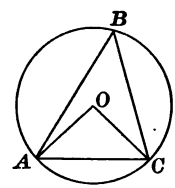
When the arc subtended by a chord is spoken of, that arc which is less than a semi-circumference is understood, unless the contrary is specified.

A *segment* of a circle is the portion included between an arc and its chord; as *AMBN*.

A *semicircle* is a segment equal to one-half the circle.

A *sector* of a circle is the portion included between an arc and the radii drawn to its extremities; as *OCD*.

148. A *central angle* is an angle whose vertex is at the centre, and whose sides are radii; as *AOC*.

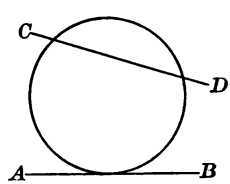


An *inscribed angle* is an angle whose vertex is on the circumference, and whose sides are chords; as *ABC*.

An angle is said to be *inscribed in a segment* when its vertex is on the arc of the segment, and its sides pass through the extremities of the subtending chord.

Thus, the angle *B* is inscribed in the segment *ABC*.

149. A straight line is said to *touch*, or be *tangent to, a circle* when it has but one point in common with the circumference; as *AB*.



In such a case, the circle is said to be tangent to the straight line.

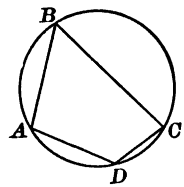
The common point is called the *point of contact*, or *point of tangency*.

A *secant* is a straight line which intersects the circumference in two points; as *CD*.

150. Two circles are said to be *tangent to each other* when they are both tangent to the same straight line at the same point.

They are said to be tangent *internally* or *externally* according as one circle lies entirely within or entirely without the other.

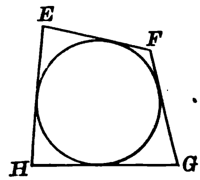
A *common tangent* to two circles is a straight line which is tangent to both of them.



151. A polygon is said to be *inscribed in a circle* when its vertices lie on the circumference; as *ABCD*.

In such a case, the circle is said to be *circumscribed about the polygon.*

A polygon is said to be *inscriptible* when it can be inscribed in a circle.

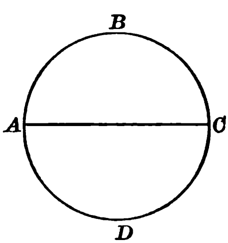


A polygon is said to be *circumscribed about a circle* when its sides are tangent to the circle; as *EFGH*.

In such a case, the circle is said to be *inscribed in the polygon.*

PROPOSITION I. THEOREM.

152. *Every diameter bisects the circle and its circumference.*



Let AC be a diameter of the circle $ABCD$.

To prove that AC bisects the circle and its circumference.

Superpose the segment ABC upon ADC, by folding it over about AC as an axis.

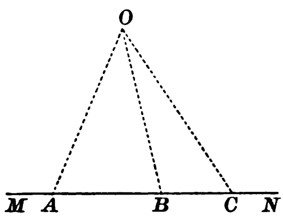
Then, the arc ABC will coincide with the arc ADC; for otherwise there would be points of the circumference unequally distant from the centre.

Hence, the segments ABC and ADC coincide throughout, and are equal.

Therefore, AC bisects the circle and its circumference.

Proposition II. Theorem.

153. *A straight line cannot intersect a circumference in more than two points.*



Let O be the centre of a circle, and MN any straight line.

To prove that MN cannot intersect the circumference in more than two points.

If possible, let MN intersect the circumference in three points, A, B, and C; and draw OA, OB, and OC.

Then, $OA = OB = OC$. (§ 143.)

We should then have three equal straight lines drawn from a point to a straight line, which is impossible. (§ 49.)

Therefore, MN cannot intersect the circumference in more than two points.

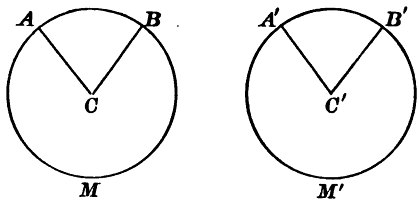
EXERCISES.

1. What is the locus of points at a given distance from a given point?

2. If two circles intersect each other, the distance between their centres is greater than the difference of their radii. (§ 62.)

Proposition III. Theorem.

154. *In equal circles, or in the same circle, equal central angles intercept equal arcs on the circumference.*



Let C and C' be the centres of the equal circles AMB and $A'M'B'$; and let $\angle ACB = \angle A'C'B'$.

To prove arc $AB =$ arc $A'B'$.

Superpose the sector ABC upon $A'B'C'$ in such a way that $\angle C$ shall coincide with its equal $\angle C'$.

Now, $AC = A'C'$, and $BC = B'C'$. (§ 145.)

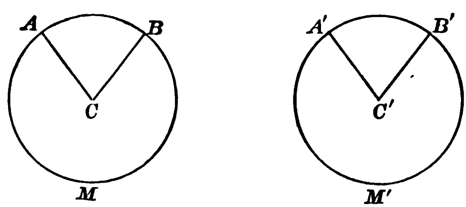
Whence, the point A will fall at A', and B at B'.

Then, the arc AB will coincide with the arc $A'B'$; for all points in each are equally distant from the centre.

Therefore, arc $AB =$ arc $A'B'$.

Proposition IV. Theorem.

155. (Converse of Prop. III.) *In equal circles, or in the same circle, equal arcs are intercepted by equal central angles.*



Let C and C' be the centres of the equal circles AMB and $A'M'B'$; and let arc $AB =$ arc $A'B'$.

To prove $\angle ACB = \angle A'C'B'$.

Since the circles are equal, we may superpose AMB upon $A'M'B'$ in such a way that the point A shall fall at A'; the centre C falling at C'.

Then since arc AB = arc $A'B'$, the point B will fall at B'.

Whence, the radii AC and BC will coincide with $A'C'$ and $B'C'$, respectively. (Ax. 5.)

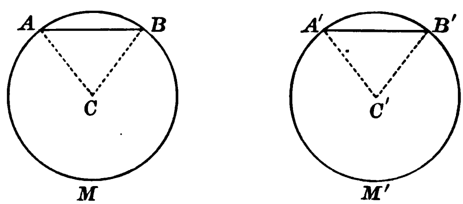
Therefore, $\angle ACB$ will coincide with $\angle A'C'B'$.

Whence, $\angle ACB = \angle A'C'B'$.

156. Sch. In equal circles, or in the same circle, the greater of two central angles intercepts the greater arc on the circumference; and conversely.

Proposition V. Theorem.

157. *In equal circles, or in the same circle, equal chords subtend equal arcs.*



In the equal circles AMB and $A'M'B'$, let

chord AB = chord $A'B'$.

To prove arc AB = arc $A'B'$.

Let C and C' be the centres of the circles; and draw AC, BC, $A'C'$, and $B'C'$.

Then in the triangles ABC and $A'B'C'$, by hypothesis,

$$AB = A'B'.$$

Also, $AC = A'C'$, and $BC = B'C'$ (§ 145.)

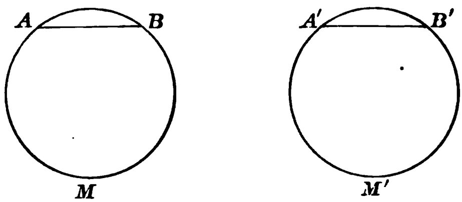
Therefore, $\triangle ABC = \triangle A'B'C'$. (§ 69.)

Whence, $\angle C = \angle C'$. (§ 66.)

Therefore, arc AB = arc $A'B'$. (§ 154.)

Proposition VI. Theorem.

158. (Converse of Prop. V.) *In equal circles, or in the same circle, equal arcs are subtended by equal chords.*



In the equal circles AMB and $A'M'B'$, let

$$\text{arc } AB = \text{arc } A'B'.$$

To prove $\quad$ chord AB = chord $A'B'$.

Since the circles are equal, we may superpose AMB upon $A'M'B'$ in such a way that the point A shall fall at A'.

Then since arc AB = arc $A'B'$, the point B will fall at B'.

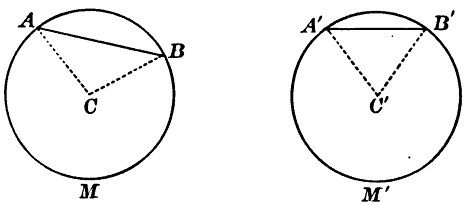
Therefore, chord AB will coincide with chord $A'B'$.

(Ax. 5.)

Whence, $\quad$ chord AB = chord $A'B'$.

Proposition VII. Theorem.

159. *In equal circles, or in the same circle, the greater of two arcs is subtended by the greater chord; each arc being less than a semi-circumference.*



Let AMB and $A'M'B'$ be equal circles.

Let arc AB be greater than arc $A'B'$; each arc being less than a semi-circumference.

To prove chord $AB >$ chord $A'B'$.

Let C and C' be the centres of the circles; and draw AC, BC, $A'C'$, and $B'C'$.

Then in the triangles ABC and $A'B'C'$,

$AC = A'C'$, and $BC = B'C'$. (§ 145.)

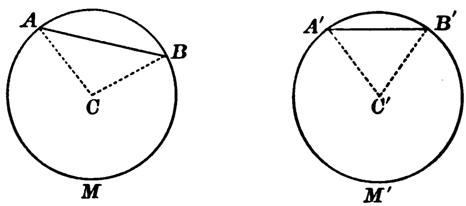
And since arc $AB >$ arc $A'B'$,

$\angle C > \angle C'$. (§ 156.)

Therefore, chord $AB >$ chord $A'B'$. (§ 89.)

Proposition VIII. Theorem.

160. (Converse of Prop. VII.) *In equal circles, or in the same circle, the greater of two chords subtends the greater arc; each arc being less than a semi-circumference.*



In the equal circles AMB and $A'M'B'$, let chord AB be greater than chord $A'B'$.

To prove arc $AB >$ arc $A'B'$;
each arc being less than a semi-circumference.

Let C and C' be the centres of the circles; and draw AC, BC, $A'C'$, and $B'C'$.

Then in the triangles ABC and $A'B'C'$,

$AC = A'C'$, and $BC = B'C'$. (§ 145.)

And by hypothesis, $AB > A'B'$.

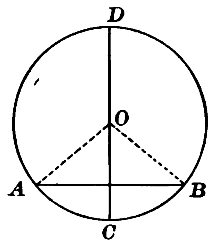
Therefore, $\angle C > \angle C'$. (§ 90.)

Whence, arc $AB >$ arc $A'B'$. (§ 156.)

161. Sch. If each arc is *greater* than a semi-circumference, the greater arc is subtended by the less chord; and conversely the greater chord subtends the less arc.

Proposition IX. Theorem.

162. *The diameter perpendicular to a chord bisects the chord and its subtended arcs.*



In the circle ADB, let the diameter CD be perpendicular to the chord AB.

To prove that CD bisects AB, and the arcs ACB and ADB.

Let O be the centre of the circle, and draw OA and OB.

Then, $OA = OB$. (§ 143.)

Hence, the triangle OAB is isosceles.

Therefore, CD bisects AB, and the angle AOB. (§ 92.)

Whence, $\angle AOC = \angle BOC$.

Therefore, arc AC = arc BC. (§ 154.)

But, arc CAD = arc CBD. (§ 152.)

Whence,

$$\text{arc } CAD - \text{arc } AC = \text{arc } CBD - \text{arc } BC.$$

That is, arc AD = arc BD.

Hence, CD bisects AB, and the arcs ACB and ADB.

163. Cor. *The perpendicular erected at the middle point of a chord passes through the centre of the circle, and bisects the arcs subtended by the chord.*

EXERCISES.

3. The diameter which bisects a chord is perpendicular to it and bisects its subtended arcs.

4. The diameter which bisects an arc bisects its chord at right angles.

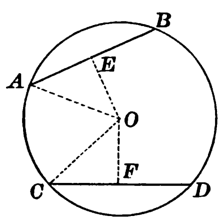
5. The straight line which bisects the arcs subtended by a chord bisects the chord at right angles.

6. The straight line which bisects a chord and its subtended arc is perpendicular to the chord.

7. The perpendiculars to the sides of an inscribed quadrilateral at their middle points meet in a common point.

Proposition X. Theorem.

164. *In the same circle, or in equal circles, equal chords are equally distant from the centre.*



Let AB and CD be equal chords of the circle ABD.

To prove AB and CD equally distant from the centre O.

Draw OE and OF perpendicular to AB and CD, respectively, and draw OA and OC.

Then E is the middle point of AB, and F of CD. (§ 162.)

Now in the right triangles OAE and OCF,

$$AE = CF,$$

being halves of equal chords.

Also, $OA = OC$. (§ 143.)

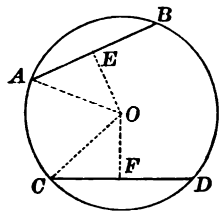
Therefore, $\triangle OAE = \triangle OCF$. (§ 88.)

Whence, $OE = OF$. (§ 66.)

Then AB and CD are equally distant from O.

PROPOSITION XI. THEOREM.

165. (Converse of Prop. X.) *In the same circle, or in equal circles, chords equally distant from the centre are equal.*



Let O be the centre of the circle ABD; and draw OE and OF perpendicular to AB and CD, respectively.

To prove that if $OE = OF$, chord AB = chord CD.

Draw OA and OC; then in the right triangles OAE and OCF,

$OA = OC$. (§ 143.)

And by hypothesis, $OE = OF$.

Therefore, $\triangle OAE = \triangle OCF$. (§ 88.)

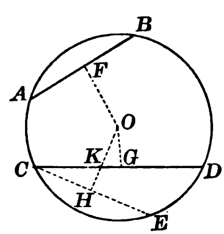
Whence, $AE = CF$. (§ 66.)

But E is the middle point of AB, and F of CD. (§ 162.)

Therefore, $2\,AE = 2\,CF$, or $AB = CD$.

PROPOSITION XII. THEOREM.

166. *In the same circle, or in equal circles, the less of two chords is at the greater distance from the centre.*



In the circle ABD, let chord AB be less than chord CD.

Draw OF and OG perpendicular to AB and CD, respectively.

To prove $OF > OG$.

Since chord $AB <$ chord CD, arc $AB <$ arc CD. (§ 160.)

Lay off arc $CE =$ arc AB, and draw CE.

Then, chord $CE =$ chord AB. (§ 158.)

Draw OH perpendicular to CE, intersecting CD at K.

Then $OH = OF$. (§ 164.)

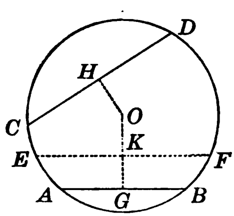
But, $OH > OK$.

And, $OK > OG$. (§ 45.)

Whence, OH, or its equal OF, is greater than OG.

Proposition XIII. Theorem.

167. (Converse of Prop. XII.) *In the same circle, or in equal circles, if two chords are unequally distant from the centre, the more remote is the less.*



In the circle ABD, let chord AB be more remote from the centre O than chord CD.

To prove chord $AB <$ chord CD.

Draw OG and OH perpendicular to AB and CD, respectively, and on OG lay off $OK = OH$.

Through K draw the chord EF perpendicular to OK.

Then, chord $EF =$ chord CD. (§ 165.)

Now, chord AB is parallel to chord EF. (§ 54.)

Whence, arc $AB <$ arc EF.

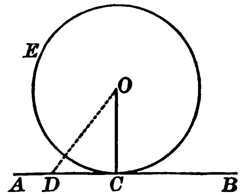
Therefore, chord $AB <$ chord EF. (§ 159.)

Whence, chord $AB <$ chord CD.

168. Cor. *A diameter of a circle is greater than any other chord; for a chord which passes through the centre is greater than any chord which does not.* (§ 167.)

Proposition XIV. Theorem.

169. *A straight line perpendicular to a radius of a circle at its extremity is tangent to the circle.*



Let the line AB be perpendicular to the radius OC of the circle EC, at its extremity C.

To prove AB tangent to the circle.

Let OD be any other straight line drawn from O to AB.

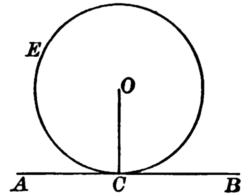
Then, $OD > OC$. (§ 45.)

Therefore, the point D lies without the circle.

Then every point of AB except C lies without the circle, and AB is tangent to the circle. (§ 149.)

Proposition XV. Theorem.

170. (Converse of Prop. XIV.) *A tangent to a circle is perpendicular to the radius drawn to the point of contact.*



Let the line AB be tangent to the circle EC.

To prove that AB is perpendicular to the radius OC drawn to the point of contact.

If AB is tangent to the circle at C, every point of AB except C lies without the circle.

Then OC is the shortest line that can be drawn from O to AB.

Therefore, OC is perpendicular to AB. (§ 45.)

171. COR. *A line perpendicular to a tangent at its point of contact passes through the centre of the circle.*

EXERCISES.

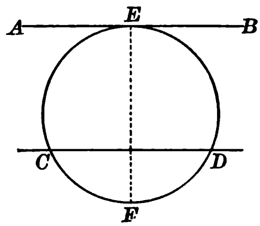
8. The tangents to a circle at the extremities of a diameter are parallel.

9. If two circles are concentric, any two chords of the greater which are tangent to the less are equal. (§ 165.)

PROPOSITION XVI. THEOREM.

172. *Two parallels intercept equal arcs on a circumference.*

CASE I. *When one line is a tangent and the other a secant.*



Let AB be tangent to the circle CED at E; and let CD be a secant parallel to AB.

To prove arc CE = arc DE.

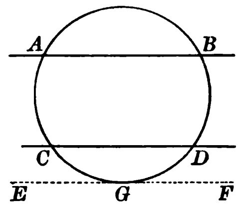
Draw the diameter EF.

Then EF is perpendicular to AB. (§ 170.)

It is therefore perpendicular to CD. (§ 56.)

Whence, arc CE = arc DE. (§ 162.)

CASE II. *When both lines are secants.*



In the circle ABD, let AB and CD be parallel secants.

To prove arc AC = arc BD.

Draw EF parallel to AB, tangent to the circle at G.

Then EF is parallel to CD. (§ 55.)

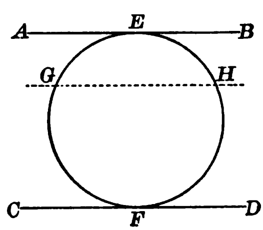
Now, arc AG = arc BG,

and arc CG = arc DG. (§ 172, Case I.)

Subtracting, we have

arc AC = arc BD.

CASE III. *When both lines are tangents.*



In the circle EF, let the lines AB and CD be parallel, and tangent to the circle at E and F, respectively.

To prove arc EGF = arc EHF.

Draw the secant GH parallel to AB.

Then GH is parallel to CD. (§ 55.)

Now, arc EG = arc EH,

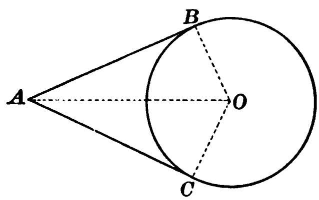
and arc FG = arc FH. (§ 172, Case I.)

Adding, we have arc EGF = arc EHF.

173. Cor. *The straight line joining the points of contact of two parallel tangents is a diameter.*

Proposition XVII. Theorem.

174. *The two tangents to a circle from an outside point are equal.*



Let AB and AC be tangent at the points B and C, respectively, to the circle whose centre is O.

To prove $$AB = AC.$$

Draw OA, OB, and OC.

Then OB and OC are perpendicular to AB and AC, respectively. (§ 170.)

Now in the right triangles OAB and OAC, OA is common.

Also, $OB = OC$. (§ 143.)

Therefore, $\triangle OAB = \triangle OAC$. (§ 88.)

Whence, $AB = AC$. (§ 66.)

175. Cor. From the equal triangles OAB and OAC, we have

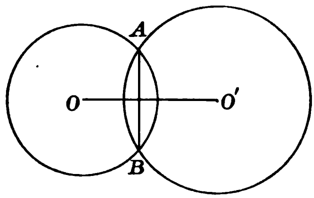
$$\angle OAB = \angle OAC, \text{ and } \angle AOB = \angle AOC.$$

That is, *the line joining the centre of a circle to the point of intersection of two tangents, bisects the angle between the tangents, and also bisects the angle between the radii drawn to the points of contact.*

Ex. 10. The straight line drawn from the centre of a circle to the point of intersection of two tangents bisects at right angles the chord joining their points of contact.

PROPOSITION XVIII. THEOREM.

176. *If two circumferences intersect, the straight line joining their centres bisects their common chord at right angles.*



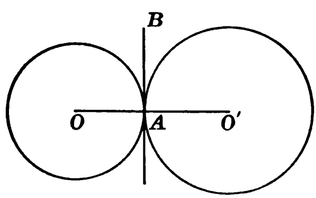
Let O and O' be the centres of two circles whose circumferences intersect at A and B; and draw OO' and AB.

To prove that OO' bisects AB at right angles.

The point O is equally distant from A and B. (§ 143.)
In like manner, O' is equally distant from A and B.
Therefore, OO' bisects AB at right angles. (§ 43.)

PROPOSITION XIX. THEOREM.

177. *If two circles are tangent to each other, the straight line joining their centres passes through their point of contact.*



Let O and O' be the centres of two circles, which are tangent to the straight line AB at A (§ 150).

To prove that the straight line joining O and O' passes through A.

Draw the radii OA and $O'A$.

Then OA and $O'A$ are perpendicular to AB. (§ 170.)

Hence, OA and $O'A$ lie in the same straight line. (§ 38.)

But only one straight line can be drawn between O and O'. (Ax. 5.)

Therefore, the straight line joining O and O' passes through A.

EXERCISES.

11. If the inscribed and circumscribed circles of a triangle are concentric, prove that the triangle is equilateral.

12. Two intersecting chords which make equal angles with the diameter passing through their point of intersection are equal. (§ 165.)

13. In an inscribed trapezoid, the non-parallel sides are equal, and also the diagonals. (§ 158.)

14. If AB and AC are the tangents from the point A to the circle whose centre is O, prove that $\angle BAC = 2 \angle OBC$.

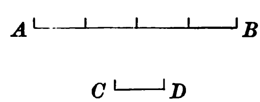
ON MEASUREMENT.

178. *Ratio* is the relation with respect to magnitude which one quantity bears to another of the same kind; and is expressed by writing the first quantity as the numerator, and the second as the denominator, of a fraction.

Thus, if a and b are quantities of the same kind, the ratio of a to b is expressed $\frac{a}{b}$; it may also be expressed $a : b$.

179. A geometrical magnitude is *measured* by finding its ratio to another magnitude of the same kind, called the *unit of measure.*

Thus, the measure of the line AB,

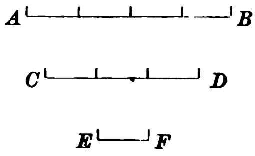


referred to the line CD as the unit, is $\frac{AB}{CD}$.

If the quotient can be obtained exactly as an integer or fraction, the result is called the *numerical measure* of the given magnitude.

Thus, if CD is contained 4 times in AB, the numerical measure of AB, with respect to CD as the unit, is 4.

180. Two geometrical magnitudes of the same kind are said to be *commensurable* when there is another magnitude of the same kind which is contained an *integral* number of times in each.



Thus, if the line EF is contained 4 times in the line AB, and 3 times in the line CD, AB and CD are commensurable.

Again, two lines whose lengths are $2\frac{3}{4}$ inches and $3\frac{4}{5}$ inches are commensurable; for the unit of measure $\frac{1}{20}$ inch is contained an integral number of times in each; i.e., 55 times in the first line, and 76 times in the second.

181. Two geometrical magnitudes of the same kind are said to be *incommensurable* when no magnitude of the same kind can be found which is contained an integral number of times in each.

For example, let AB and CD be two lines such that

$$\frac{AB}{CD} = \sqrt{2}.$$

Since the value of $\sqrt{2}$ can only be obtained *approximately* as a decimal, it is evident that no line, *however small*, can be found which is contained an integral number of times in each line.

Then AB and CD are incommensurable.

182. Although a magnitude which is incommensurable with respect to the unit has, strictly speaking, no numerical measure (§ 179), still if CD is the unit of measure, and $\frac{AB}{CD} = \sqrt{2}$, we shall speak of $\sqrt{2}$ as the numerical measure of AB.

183. It is evident from the above that the ratio of two magnitudes of the same kind, whether commensurable or incommensurable, is equal to the ratio of their numerical measures when referred to a common unit.

THE METHOD OF LIMITS.

184. A *variable quantity*, or simply a *variable*, is a quantity which may assume, under the conditions imposed upon it, an indefinitely great number of *different* values.

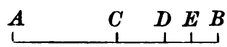
185. A *constant* is a quantity which remains unchanged throughout the same discussion.

186. A *limit* of a variable is a constant quantity, the difference between which and the variable may be made less than any assigned quantity, however small, without ever becoming zero.

In other words, a limit of a variable is a fixed quantity which the variable may approach as near as we please, but can never actually reach.

The variable is said to *approach* its limit.

187. Suppose, for example, that a point moves from A towards B under the condition that it shall move, during successive equal intervals of time, first from A to C, half-way from A to B; then to D, half-way from C to B; then to E, half-way from D to B; and so on indefinitely.



In this case, the distance from A to the moving point is a variable which approaches the constant value AB as a limit; for the distance between the moving point and B can be made less than any assigned quantity, however small, but cannot be made actually equal to zero.

Again, the distance from B to the moving point is a variable which approaches the limit 0.

As another illustration, consider the series

$$1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}, \cdots,$$

where each term after the first is one-half the preceding.

In this case, by taking terms enough, the last term may be made less than any assigned number, however small, but cannot be made actually equal to 0.

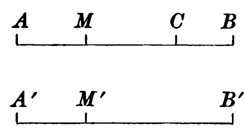
Then, the last term of the series is a variable which approaches the limit 0 when the number of terms is indefinitely increased.

Again, the sum of the first two terms is $1\frac{1}{2}$;
the sum of the first three terms is $1\frac{3}{4}$;
the sum of the first four terms is $1\frac{7}{8}$; etc.

In this case, by taking terms enough, the sum of the terms may be made to differ from 2 by less than any assigned number, however small, but cannot be made actually equal to 2.

Then, the sum of the terms of the series is a variable which approaches the limit 2 when the number of terms is indefinitely increased.

188. THE THEOREM OF LIMITS. *If two variables are always equal, and each approaches a limit, the limits are equal.*



Let AM and $A'M'$ be two variables, which are always equal, and approach the limits AB and $A'B'$, respectively.

To prove $AB = A'B'$.

If possible, let AB be greater than $A'B'$, and lay off $AC = A'B'$.

Then the variable AM may assume values between AC and AB, while the variable $A'M'$ is restricted to values less than AC; which is contrary to the hypothesis that the variables are always equal.

Hence, AB cannot be greater than $A'B'$.

In like manner, it may be proved that AB cannot be less than $A'B'$.

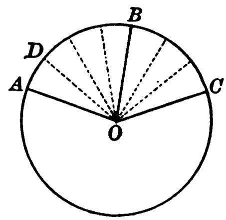
Therefore, $AB = A'B'$.

MEASUREMENT OF ANGLES.

Proposition XX. Theorem.

189. *In the same circle, or in equal circles, two central angles are in the same ratio as their intercepted arcs.*

Case I. *When the arcs are commensurable* (§ 180).



In the circle ABC, let AOB and BOC be central angles, intercepting the commensurable arcs AB and BC, respectively.

To prove $$\frac{\angle AOB}{\angle BOC} = \frac{\text{arc } AB}{\text{arc } BC}.$$

Let AD be a common measure of the arcs AB and BC; and suppose it to be contained 4 times in AB, and 3 times in BC.

Then, $$\frac{\text{arc } AB}{\text{arc } BC} = \frac{4}{3}. \quad (1)$$

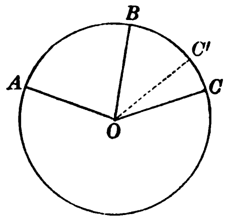
Drawing radii to the several points of division, $\angle AOB$ will be divided into 4 parts, and $\angle BOC$ into 3 parts, all of which parts will be equal. (§ 155.)

Whence, $$\frac{\angle AOB}{\angle BOC} = \frac{4}{3}. \quad (2)$$

From (1) and (2), we have

$$\frac{\angle AOB}{\angle BOC} = \frac{\text{arc } AB}{\text{arc } BC}. \quad \text{(Ax. 1.)}$$

CASE II. *When the arcs are incommensurable* (§ 181).



In the circle ABC, let AOB and BOC be central angles, intercepting the incommensurable arcs AB and BC, respectively.

To prove $$\frac{\angle AOB}{\angle BOC} = \frac{\text{arc } AB}{\text{arc } BC}.$$

Let the arc AB be divided into any number of equal parts, and let one of these parts be applied to the arc BC as a measure.

Since AB and BC are incommensurable, a certain number of the parts will extend from B to C', leaving a remainder CC' less than one of the parts.

Draw OC'.

Then since the arcs AB and BC' are commensurable, we have

$$\frac{\angle AOB}{\angle BOC'} = \frac{\text{arc } AB}{\text{arc } BC'}. \quad \text{(§ 189, Case I.)}$$

Now let the number of subdivisions of the arc AB be indefinitely increased.

Then the length of each part will be indefinitely diminished; and the remainder CC', being always less than one of the parts, will approach the limit 0.

Then, $\dfrac{\angle AOB}{\angle BOC'}$ will approach the limit $\dfrac{\angle AOB}{\angle BOC}$,

and, $\dfrac{\text{arc } AB}{\text{arc } BC'}$ will approach the limit $\dfrac{\text{arc } AB}{\text{arc } BC}$.

Now, $\dfrac{\angle AOB}{\angle BOC'}$ and $\dfrac{\text{arc } AB}{\text{arc } BC'}$ are two variables which are always equal, and approach the limits $\dfrac{\angle AOB}{\angle BOC}$ and $\dfrac{\text{arc} AB}{\text{arc} BC}$, respectively.

By the Theorem of Limits, these limits are equal. (§ 188.)

Therefore, $$\frac{\angle AOB}{\angle BOC} = \frac{\text{arc} AB}{\text{arc} BC}.$$

190. Sch. The usual unit of measure for arcs is the *degree*, which is the ninetieth part of a quadrant (§ 146).

The degree of arc is divided into sixty equal parts, called *minutes*, and the minute into sixty equal parts, called *seconds*.

If the sum of two arcs is a quadrant, or 90°, one is called the *complement* of the other; if their sum is a semi-circumference, or 180°, one is called the *supplement* of the other.

191. Cor. I. By § 154, equal central angles, in the same circle, intercept equal arcs on the circumference.

Hence, if the angular magnitude about the centre of a circle be divided into four equal parts, each part will intercept one-fourth of the circumference; that is,

A right central angle intercepts a quadrant on the circumference.

192. Cor. II. By § 189, a central angle of n degrees bears the same ratio to a right central angle as its intercepted arc bears to a quadrant.

But a central angle of n degrees is $\frac{n}{90}$ of a right central angle.

Hence, its intercepted arc is $\frac{n}{90}$ of a quadrant, or an arc of n degrees.

The above principle is usually expressed as follows:

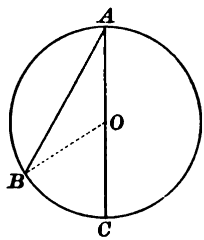
A central angle is measured by its intercepted arc.

This means simply that the number of degrees in the angle is equal to the number of degrees in its intercepted arc.

Proposition XXI. Theorem.

193. *An inscribed angle is measured by one-half its intercepted arc.*

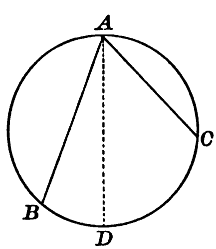
Case I. *When one side of the angle is a diameter.*



Let AC be a diameter, and AB a chord, of the circle ABC.
To prove that $\angle BAC$ is measured by $\frac{1}{2}$ arc BC.

Draw the radius OB; then, $OA = OB$. (§ 143.)
Whence, $\angle B = \angle A$. (§ 91.)
But, $\angle BOC = \angle A + \angle B$. (§ 83, 1.)
Therefore, $\angle BOC = 2 \angle A$, or $\angle A = \frac{1}{2} \angle BOC$.
But, $\angle BOC$ is measured by arc BC. (§ 192.)
Whence, $\angle A$ is measured by $\frac{1}{2}$ arc BC.

Case II. *When the centre is within the angle.*



Let AD be a diameter of the circle ABC.
To prove that the inscribed angle BAC is measured by $\frac{1}{2}$ arc BC.

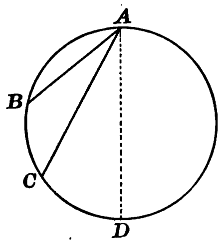
Now, $\angle BAD$ is measured by $\frac{1}{2}$ arc BD,
and $\angle CAD$ is measured by $\frac{1}{2}$ arc CD.

(§ 193, Case I.)

Therefore, the sum of the angles BAD and CAD is measured by one-half the sum of the arcs BD and CD.

Hence, $\angle BAC$ is measured by $\frac{1}{2}$ arc BC.

CASE III. *When the centre is without the angle.*



Let AD be a diameter of the circle ABC.

To prove that the inscribed angle BAC is measured by $\frac{1}{2}$ arc BC.

Now, $\angle BAD$ is measured by $\frac{1}{2}$ arc BD,
and $\angle CAD$ is measured by $\frac{1}{2}$ arc CD.

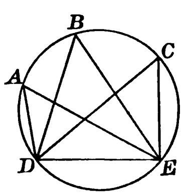
(§ 193, Case I.)

Therefore, the difference of the angles BAD and CAD is measured by one-half the difference of the arcs BD and CD.

Hence, $\angle BAC$ is measured by $\frac{1}{2}$ arc BC.

194. SCH. As explained in § 192, this proposition means simply that the number of degrees in an inscribed angle is one-half the number of degrees in its intercepted arc.

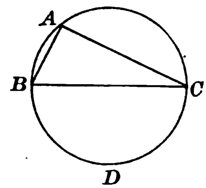
195. COR. I. *All angles inscribed in the same segment are equal.*



Thus, if the angles A, B, and C are inscribed in the segment ADE, then each angle is measured by $\frac{1}{2}$ arc DE. (§ 193.)

Whence, $\angle A = \angle B = \angle C$.

196. Cor. II. *An angle inscribed in a semicircle is a right angle.*



Let BC be a diameter of the circle ABD, and BAC an angle inscribed in the semicircle ABC.

To prove BAC a right angle.

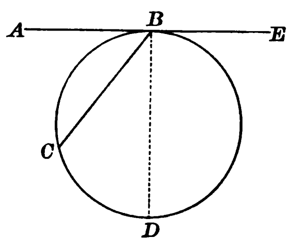
$\angle BAC$ is measured by $\frac{1}{2}$ arc BDC. (§ 193.)

But, $\frac{1}{2}$ arc BDC is a quadrant.

Whence, $\angle BAC$ is a right angle.

Proposition XXII. Theorem.

197. *The angle between a tangent and a chord is measured by one-half its intercepted arc.*



Let AE be tangent to the circle BCD at B, and let BC be a chord.

To prove that $\angle ABC$ is measured by $\frac{1}{2}$ arc BC.

Draw the diameter BD.

Then BD is perpendicular to AE. (§ 170.)

Now since a right angle is measured by one-half a semicircumference,

$\angle ABD$ is measured by $\frac{1}{2}$ arc BCD.

Also, $\angle CBD$ is measured by $\frac{1}{2}$ arc CD. (§ 193.)

Hence,

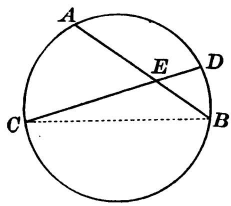
$\angle ABD - \angle CBD$ is measured by $\frac{1}{2}$ (arc BCD − arc CD).

That is, $\angle ABC$ is measured by $\frac{1}{2}$ arc BC.

In like manner, $\angle EBC$ is measured by $\frac{1}{2}$ arc BDC.

Proposition XXIII. Theorem.

198. *The angle between two chords, intersecting within the circumference, is measured by one-half the sum of its intercepted arc, and the arc intercepted by its vertical angle.*



In the circle ABC, let the chords AB and CD intersect at E.

To prove that

$\angle AEC$ is measured by $\frac{1}{2}$ (arc AC + arc BD).

Draw BC.

Then, $\angle AEC = \angle B + \angle C$. (§ 83, 1.)

But, $\angle B$ is measured by $\frac{1}{2}$ arc AC,

and $\angle C$ is measured by $\frac{1}{2}$ arc BD. (§ 193.)

Whence,

$\angle AEC$ is measured by $\frac{1}{2}$ (arc AC + arc BD).

EXERCISES.

15. If, in the figure of § 195, the arc DE is $\frac{7}{24}$ of the circumference, how many degrees are there in the angle A?

16. If, in the figure of § 197, arc $BC = 107°$, how many degrees are there in the angles ABC and EBC?

17. If, in the figure of § 197, $\angle ABC = 42°$, how many degrees are there in the arc CD?

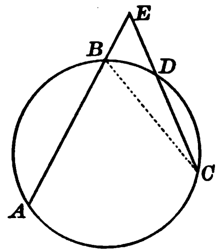
18. If, in the above figure, arc $AD = 94°$, and $\angle AEC = 51°$, how many degrees are there in the arc BC?

19. Prove Prop. XXII. by drawing a radius perpendicular to BC.

20. Prove Prop. XXIII. by drawing through B a chord parallel to CD.

Proposition XXIV. Theorem.

199. *The angle between two secants, intersecting without the circumference, is measured by one-half the difference of the intercepted arcs.*



In the circle ABC, let the secants AE and CE intersect the circumference in the points A and B, and C and D, respectively.

To prove that $\angle E$ is measured by $\frac{1}{2}$ (arc AC — arc BD).

Draw BC.

Then, $\angle ABC = \angle E + \angle C$. (§ 83, 1.)

Whence, $\angle E = \angle ABC - \angle C$.

But, $\angle ABC$ is measured by $\frac{1}{2}$ arc AC,

and $\angle C$ is measured by $\frac{1}{2}$ arc BD. (§ 193.)

Whence, $\angle E$ is measured by $\frac{1}{2}$ (arc AC — arc BD).

EXERCISES.

21. If, in the above figure, the arcs AC and BD are respectively two-ninths and one-sixteenth of the circumference, how many degrees are there in the angle E?

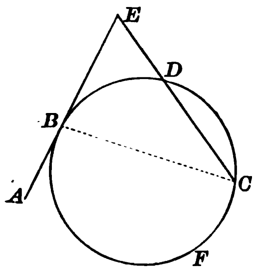
22. If, in the above figure, arc $AC = 117°$, and $\angle C = 14°$, how many degrees are there in the angle E?

23. If, in the above figure, AC is a quadrant, and $\angle E = 39°$, how many degrees are there in the arc BD?

24. Prove Prop. XXIV. by drawing through B a chord parallel to CD.

Proposition XXV. Theorem.

200. *The angle between a secant and a tangent is measured by one-half the difference of the intercepted arcs.*



Let AE be tangent to the circle BDC at B; and let EC be a secant, intersecting the circumference in the points C and D.

To prove that

$\angle E$ is measured by $\frac{1}{2}$ (arc BFC — arc BD).

Draw BC.

Then, $\angle ABC = \angle E + \angle C$. (§ 83, 1.)

Whence, $\angle E = \angle ABC - \angle C$.

But, $\angle ABC$ is measured by $\frac{1}{2}$ arc BFC. (§ 197.)

And, $\angle C$ is measured by $\frac{1}{2}$ arc BD. (§ 193.)

Whence, $\angle E$ is measured by $\frac{1}{2}$ (arc BFC — arc BD).

EXERCISES.

25. If, in the above figure, arc $BFC = 197°$, and arc $CD = 75°$, how many degrees are there in the angle E?

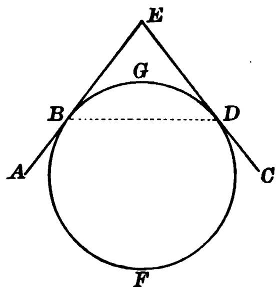
26. If, in the above figure, $\angle E = 53°$, and the arc BD is one-fifth of the circumference, how many degrees are there in the arc BFC?

27. Prove Prop. XXV. by drawing through D a chord parallel to AE.

28. If two chords intersect at right angles within the circumference of a circle, the sum of the opposite intercepted arcs is equal to a semi-circumference.

PROPOSITION XXVI. THEOREM.

201. *The angle between two tangents is measured by one-half the difference of the intercepted arcs.*



Let AE and CE be tangent to the circle BDF at B and D, respectively.

To prove that

$\angle E$ is measured by $\frac{1}{2}$ (arc BFD — arc BGD).

Draw BD.

Then, $\angle ABD = \angle E + \angle BDE$. (§ 83, 1.)

Whence, $\angle E = \angle ABD - \angle BDE$.

But, $\angle ABD$ is measured by $\frac{1}{2}$ arc BFD,

and $\angle BDE$ is measured by $\frac{1}{2}$ arc BGD. (§ 197.)

Whence,

$\angle E$ is measured by $\frac{1}{2}$ (arc BFD — arc BGD).

202. COR. We have from the above figure,

$\frac{1}{2}$ (arc BFD — arc BGD)

$= \frac{1}{2}$ ($360°$ — arc BGD — arc BGD)

$= \frac{1}{2}$ ($360°$ — 2 arc BGD)

$= 180°$ — arc BGD.

Then, $\angle E$ is measured by $180°$ — arc BGD.

Hence, *the angle between two tangents is measured by the supplement of the smaller of the two intercepted arcs.*

EXERCISES.

29. If, in the figure of § 201, the arc BFD is thirteen-sixteenths of the circumference, how many degrees are there in the angle E?

30. If AB and AC are two chords of a circle making equal angles with the tangent at A, prove that $AB = AC$.

31. From a given point within a circle and not coincident with the centre, not more than two equal straight lines can be drawn to the circumference. (§ 163.)

32. The sum of two opposite sides of a circumscribed quadrilateral is equal to the sum of the other two sides. (§ 174.)

33. In a circumscribed trapezoid, the straight line joining the middle points of the non-parallel sides is equal to one-fourth the perimeter of the trapezoid. (§ 132.)

34. If the opposite sides of a circumscribed quadrilateral are parallel, the figure is a rhombus or a square.

35. If tangents be drawn to a circle at the extremities of any pair of diameters which are not perpendicular to each other, the figure formed is a rhombus.

36. If the angles of a circumscribed quadrilateral are right angles, the figure is a square.

37. If two circles are tangent to each other at the point A, the tangents to them from any point in the common tangent which passes through A are equal. (§ 174.)

38. If two circles are tangent to each other externally at the point A, the common tangent which passes through A bisects the other two common tangents.

39. The bisector of the angle between two tangents to a circle passes through the centre.

40. The bisectors of the angles of a circumscribed quadrilateral pass through a common point.

41. If AB is one of the non-parallel sides of a trapezoid circumscribed about a circle whose centre is C, prove that ACB is a right angle.

42. Three consecutive sides of an inscribed quadrilateral subtend arcs of 82°, 99°, and 67° respectively. Find each angle of the quadrilateral in degrees, and the angle between its diagonals.

43. An angle inscribed in a segment greater than a semicircle is acute; and an angle inscribed in a segment less than a semicircle is obtuse.

44. The opposite angles of an inscribed quadrilateral are supplementary.

45. Prove Prop. VI. by drawing radii to the extremities of the chords.

46. Prove the theorem of § 168 by drawing radii to the extremities of the chord.

47. Prove the theorem of § 202 by drawing radii to the points of contact of the tangents.

48. Prove Prop. XIII. by *Reductio ad Absurdum.*

49. If any number of angles are inscribed in the same segment, their bisectors pass through a common point. (§ 193.)

50. Two chords perpendicular to a third chord at its extremities are equal.

51. If two opposite sides of an inscribed quadrilateral are equal and parallel, the figure is a rectangle.

52. If the diagonals of an inscribed quadrilateral intersect at the centre of the circle, the figure is a rectangle.

53. The circle described on one of the equal sides of an isosceles triangle as a diameter, bisects the base. (§ 196.)

54. If a tangent be drawn to a circle at the extremity of a chord, the middle point of the subtended arc is equally distant from the chord and from the tangent.

55. If the sides AB, BC, and CD of an inscribed quadrilateral subtend arcs of 99°, 106°, and 78° respectively, and the sides BA and CD produced meet at E, and the sides AD and BC at F, find the number of degrees in the angles AED and AFB.

56. If O is the centre of the circumscribed circle of a triangle ABC, and OD is drawn perpendicular to BC, prove that

$$\angle BOD = \angle A.$$

57. If D, E, and F are the points of contact of the sides AB, BC, and CA of a triangle circumscribed about a circle, prove that

$$\angle DEF = 90° - \tfrac{1}{2} A.$$

58. If the sides AB and BC of an inscribed quadrilateral $ABCD$ subtend arcs of 69° and 112°, and the angle AED between the diagonals is 87°, how many degrees are there in each angle of the quadrilateral?

59. If any number of parallel chords be drawn in a circle, their middle points lie in the same straight line.

60. What is the locus of the middle points of a system of parallel chords in a circle?

61. What is the locus of the middle points of a system of chords of given length in a circle?

62. If two circles are tangent to each other, any straight line drawn through their point of contact subtends arcs of the same number of degrees on their circumferences. (§ 197.)

63. If a straight line be drawn through the point of contact of two circles which are tangent to each other, terminating in their circumferences, the radii drawn to its extremities are parallel.

64. If a straight line be drawn through the point of contact of two circles which are tangent to each other, terminating in their circumferences, the tangents at its extremities are parallel.

65. If the sides AB and DC of an inscribed quadrilateral be produced to meet at E, prove that the triangles ACE and BDE, and also the triangles ADE and BCE, are mutually equiangular.

66. The sum of the angles subtended at the centre of a circle by two opposite sides of a circumscribed quadrilateral is equal to two right angles.

67. If a circle be described on the radius of another circle as a diameter, any chord of the greater passing through the point of contact of the circles is bisected by the circumference of the smaller. (§ 196.)

68. If a circle is inscribed in a right triangle, the sum of its diameter and the hypotenuse is equal to the sum of the legs.

69. If the sides AB and CD of a quadrilateral inscribed in a circle make equal angles with the diameter passing through their point of intersection, prove that $AB = CD$. (§ 165.)

70. If the angles A, B, and C of a circumscribed quadrilateral $ABCD$ are 128°, 67°, and 112°, and the sides AB, BC, CD, and DA are tangent to the circle at the points E, F, G, and H, find the number of degrees in each angle of the quadrilateral $EFGH$.

71. The least chord which can be drawn through a given point within a circle is perpendicular to the diameter passing through that point. (§ 165.)

72. If D, E, and F are the middle points of the arcs subtended by the sides AB, BC, and CA of an inscribed triangle, prove that the sum of the angles ADB, BEC, and CFA is equal to four right angles.

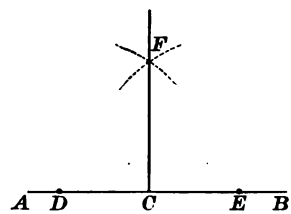
NOTE. For additional exercises on Book II., see p. 222.

PROBLEMS IN CONSTRUCTION.

Proposition XXVII. Problem.

203. *At a given point in a straight line to erect a perpendicular to that line.*

First Method.



Let C be the given point in the straight line AB.
To erect a perpendicular to AB at C.

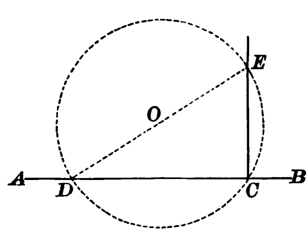
Lay off the equal distances CD and CE.

With D and E as centres, and with equal radii, describe arcs intersecting at F, and draw CF.

Then CF is perpendicular to AB at C.

For since C and F are each equally distant from D and E, CF is perpendicular to DE at its middle point. (§ 43.)

Second Method.



Let C be the given point in the straight line AB.
To erect a perpendicular to AB at C.

With any point O, without the line AB, as a centre, and the distance OC as a radius, describe a circumference intersecting AB at C and D.

Draw the diameter DE; also, draw CE.

Then CE is perpendicular to AB at C.

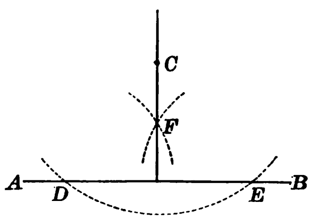
For $\angle DCE$, being inscribed in a semicircle, is a right angle. (§ 196.)

Whence, CE is perpendicular to CD.

Note. The second method of construction is preferable when the given point is near the end of the line.

Proposition XXVIII. Problem.

204. *From a given point without a straight line to draw a perpendicular to that line.*



Let C be the given point without the straight line AB.

To draw a perpendicular from C to AB.

With C as a centre, and with any convenient radius, describe an arc intersecting AB at D and E.

With D and E as centres, and with equal radii, describe arcs intersecting at F, and draw CF.

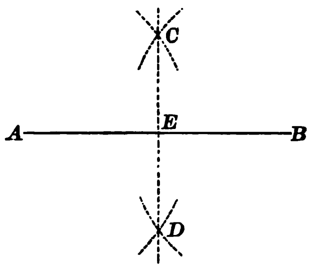
Then CF is perpendicular to AB.

For since C and F are each equally distant from D and E, CF is perpendicular to DE at its middle point. (§ 43.)

Ex. 73. Given the base and altitude of an isosceles triangle, to construct the triangle.

PROPOSITION XXIX. PROBLEM.

205. *To bisect a given straight line.*



Let AB be the given straight line.

To bisect AB.

With A and B as centres, and with equal radii, describe arcs intersecting at C and D.

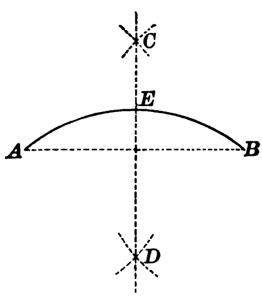
Draw CD intersecting AB at E.

Then E is the middle point of AB.

For since C and D are each equally distant from A and B, CD is perpendicular to AB at its middle point. (§ 43.)

PROPOSITION XXX. PROBLEM.

206. *To bisect a given arc.*



Let AB be the given arc.

To bisect the arc AB.

With A and B as centres, and with equal radii, describe arcs intersecting at C and D.

Draw CD intersecting the arc AB at E.

Then E is the middle point of the arc AB.

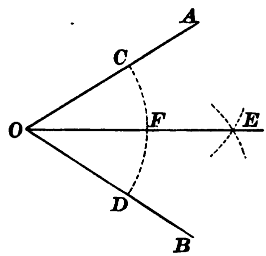
For, draw the chord AB.

Then CD is perpendicular to the chord AB at its middle point. (§ 43.)

Whence, CD bisects the arc AB. (§ 163.)

Proposition XXXI. Problem.

207. *To bisect a given angle.*



Let AOB be the given angle.

To bisect $\angle AOB$.

With O as a centre, and with any convenient radius, describe an arc intersecting OA at C, and OB at D.

With C and D as centres, and with equal radii, describe arcs intersecting at E; and draw OE.

Then OE bisects $\angle AOB$.

For since O and E are each equally distant from C and D, OE bisects the arc CD at F. (§ 206.)

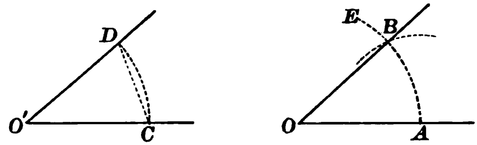
Whence, $\angle COF = \angle DOF$. (§ 155.)

That is, OE bisects $\angle AOB$.

208. Sch. By repetitions of the above construction, an angle may be divided into 4, 8, 16, etc., equal parts.

Proposition XXXII. Problem.

209. ***With a given vertex and a given side, to construct an angle equal to a given angle.***



Let O be the given vertex, OA the given side, and O' the given angle.

To construct, with O as a vertex and OA as a side, an angle equal to O'.

With O' as a centre, and with any convenient radius, describe an arc intersecting the sides of the angle O' at C and D.

With O as a centre, and with the same radius, describe the indefinite arc AE.

With A as a centre, and with a radius equal to the chord CD, describe an arc intersecting the arc AE at B; and draw OB.

Then, $\angle AOB = \angle O'$.

For by construction, the chords of the arcs AB and CD are equal.

Whence, arc AB = arc CD. (§ 157.)

Therefore, $\angle AOB = \angle O'$. (§ 155.)

EXERCISES.

74. Given a side of an equilateral triangle, to construct the triangle.

75. Given an angle, to construct its complement.

76. Given an angle, to construct its supplement.

77. To construct an angle of 60° (Ex. 74); of 30°; of 120°; of 150°.

78. To construct an angle of 45°; of 135°; of $22\frac{1}{2}$°; of $67\frac{1}{2}$°.

PROPOSITION XXXIII. PROBLEM.

210. *Given two angles of a triangle, to find the third.*

Let A and B be two angles of a triangle.
To find the third angle.

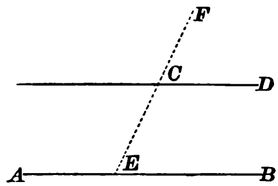
Draw the indefinite straight line CD; and at any point E construct $\angle DEF = \angle A$, and $\angle FEG = \angle B$ (§ 209).

Then CEG is the required angle.

For if two angles of a triangle are given, the third angle may be found by subtracting their sum from two right angles. (§ 82.)

PROPOSITION XXXIV. PROBLEM.

211. *Through a given point without a given straight line, to draw a parallel to the line.*



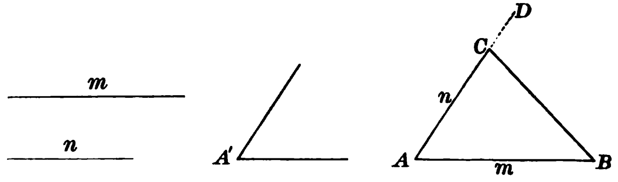
Let C be the given point without the straight line AB.
To draw through C a parallel to AB.

Through C draw any line EF, meeting AB at E.
Construct $\angle FCD = \angle CEB$ (§ 209).
Then CD is parallel to AB.

For since AB and CD are cut by EF, making the corresponding angles equal, AB and CD are parallel. (§ 76.)

Proposition XXXV. Problem.

212. *Given two sides and the included angle of a triangle, to construct the triangle.*



Let m and n be the given sides, and A' their included angle.

To construct the triangle.

Draw the straight line AB equal to m.

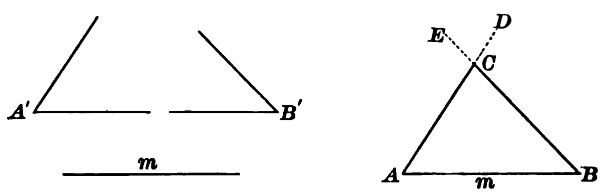
Construct $\angle BAD = \angle A'$.

On AD take $AC = n$, and draw BC.

Then ABC is the required triangle.

Proposition XXXVI. Problem.

213. *Given a side and two adjacent angles of a triangle, to construct the triangle.*



Let m be the given side, and A' and B' its adjacent angles.

To construct the triangle.

Draw the straight line AB equal to m.

Construct $\angle BAD = \angle A'$, and $\angle ABE = \angle B'$.

Let AD and BE intersect at C.

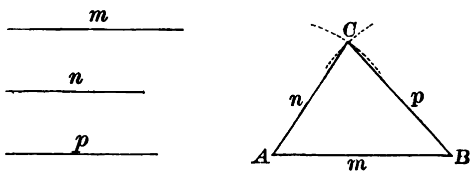
Then ABC is the required triangle.

214. Sch. I. If a side and *any* two angles of a triangle are given, the third angle may be found by § 210, and the triangle may then be constructed as in § 213.

215. Sch. II. The problem is impossible when the sum of the given angles is equal to, or greater than, two right angles. (§ 82.)

Proposition XXXVII. Problem.

216. *Given the three sides of a triangle, to construct the triangle.*



Let m, n, and p be the given sides.
To construct the triangle.

Draw the straight line AB equal to m.

With A as a centre, and with a radius equal to n, describe an arc; with B as a centre, and with a radius equal to p, describe an arc intersecting the former arc at C.

Draw AC and BC.

Then ABC is the required triangle.

217. Sch. The problem is impossible when one of the given sides is equal to, or greater than, the sum of the other two. (§ 61.)

EXERCISES.

79. Given one of the equal sides and one of the equal angles of an isosceles triangle, to construct the triangle.

80. To construct a right triangle, having given the hypotenuse and an acute angle.

81. Through a given point without a straight line to draw a line making a given angle with that line.

Proposition XXXVIII. Problem.

218. *Given two sides of a triangle, and the angle opposite to one of them, to construct the triangle.*

Let m and n be the given sides, and let A' be the angle opposite to n.

To construct the triangle.

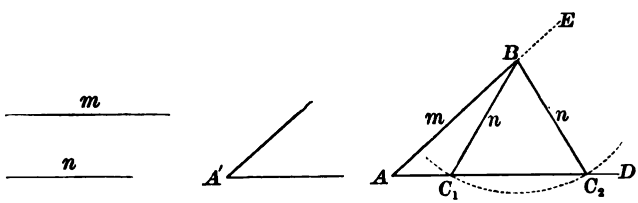
Construct $\angle DAE = \angle A'$, and on AE take $AB = m$.

With B as a centre, and with a radius equal to n, describe an arc.

Case I. *When A' is acute, and $m > n$.*

There may be three cases:

First. The arc may intersect AD in two points, C_1 and C_2.



Draw BC_1 and BC_2.

There are then two triangles, ABC_1 and ABC_2, either of which answers to the given conditions.

This is called the *ambiguous* case.

Second. The arc may be tangent to AD.

In this case there is but one solution; a *right* triangle. (§ 170.)

Third. The arc may not intersect AD at all.

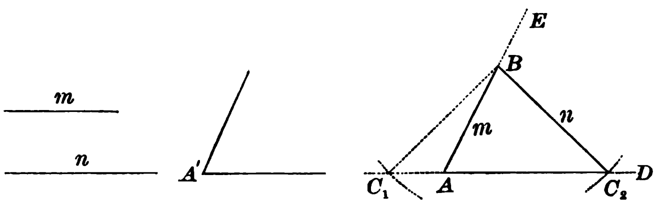
In this case the problem is impossible.

Case II. *When A' is acute, and $m = n$.*

In this case, the arc intersects AD in two points, one of which is A.

There is then but one solution; an *isosceles* triangle.

CASE III. *When A' is acute, and $m < n$.*



In this case, the arc intersects AD in two points, C_1 and C_2.

But the triangle ABC_1 does not answer to the given conditions, since it does not contain the angle A'.

There is then but one solution; the triangle ABC_2.

CASE IV. *When A' is right or obtuse, and $m < n$.*

In each of these cases, the arc intersects AD in two points on opposite sides of A.

There is then but one solution.

219. SCH. If A' is right or obtuse, and $m = n$ or $m > n$, the problem is impossible; for the side opposite the right or obtuse angle in a triangle must be the greatest side of the triangle. (§ 97.)

The student should construct the triangle corresponding to each of the cases of § 218.

EXERCISES.

82. To construct a right triangle, having given a side and the opposite acute angle.

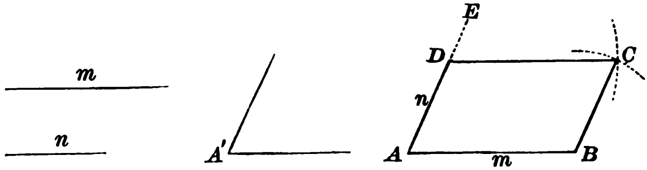
83. Given the base and the vertical angle of an isosceles triangle, to construct the triangle.

84. Given the altitude and one of the equal angles of an isosceles triangle, to construct the triangle.

85. Given two diagonals of a parallelogram and their included angle, to construct the parallelogram.

PROPOSITION XXXIX. PROBLEM.

220. *Given two sides and the included angle of a parallelogram, to construct the parallelogram.*



Let m and n be the given sides, and A' their included angle.

To construct the parallelogram.

Draw the straight line AB equal to m.

Construct $\angle BAE = \angle A'$, and on AE take $AD = n$.

With B as a centre, and with a radius equal to n, describe an arc; with D as a centre, and with a radius equal to m, describe an arc intersecting the former arc at C.

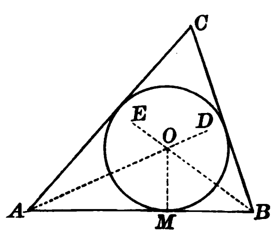
Draw BC and DC.

Then $ABCD$ is the required parallelogram.

For since, by construction, $AB = CD$ and $AD = BC$, $ABCD$ is a parallelogram. (§ 108.)

PROPOSITION XL. PROBLEM.

221. *To inscribe a circle in a given triangle.*



Let ABC be the given triangle.

To inscribe a circle in ABC.

Draw AD and BE bisecting the angles A and B, respectively (§ 207); and from their intersection O, draw OM perpendicular to AB.

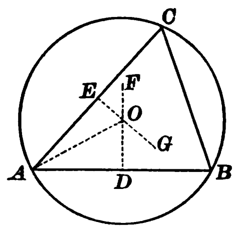
With O as a centre, and with OM as a radius, describe a circle.

This circle will be tangent to AB, BC, and CA.

For O is equally distant from AB, BC, and CA. (§ 137.)

Proposition XLI. Problem.

222. *To circumscribe a circle about a given triangle.*



Let ABC be the given triangle.

To circumscribe a circle about ABC.

Draw DF and EG perpendicular to AB and AC, respectively, at their middle points (§ 205).

Let DF and EG intersect at O.

With O as a centre, and OA as a radius, describe a circumference.

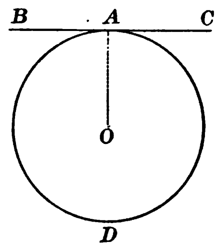
This circumference will pass through the points A, B, and C.

For O is equally distant from A, B, and C. (§ 139.)

223. Sch. The above construction serves to describe a circumference through three given points not in the same straight line, or to find the centre of a given circumference or arc.

PROPOSITION XLII. PROBLEM.

224. *To draw a tangent to a circle through a given point on the circumference.*

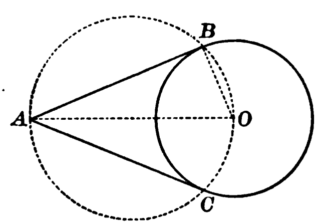


Let A be the given point on the circumference AD.
To draw through A a tangent to AD.

Draw the radius OA.
Through A draw BC perpendicular to OA (§ 203).
Then BC will be tangent to AD. (§ 169.)

PROPOSITION XLIII. PROBLEM.

225. *To draw a tangent to a circle through a given point without the circle.*



Let A be the given point without the circle BC.
To draw through A a tangent to BC.

Let O be the centre of the circle BC.
On OA as a diameter, describe a circumference, cutting the given circumference at B and C; and draw AB and AC.

Then either AB or AC is tangent to BC.

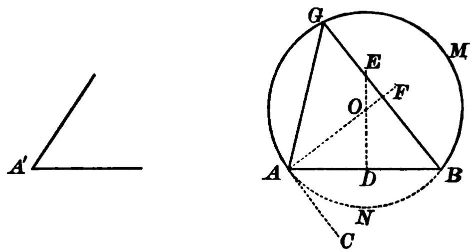
For draw OB.

Then $\angle ABO$, being inscribed in a semicircle, is a right angle. (§ 196.)

Therefore, AB is tangent to BC. (§ 169.)

Proposition XLIV. Problem.

226. *Upon a given straight line, to describe a segment which shall contain a given angle.*



Let AB be the given straight line, and A' the given angle.

To describe a circumference $AMBN$, such that every angle inscribed in the segment AMB shall be equal to A'.

Construct $\angle BAC = \angle A'$.

Draw DE perpendicular to AB at its middle point (§ 205).

Draw AF perpendicular to AC, meeting DE at O.

With O as a centre, and OA as a radius, describe the circumference $AMBN$.

Then AMB will be the required segment.

For, let AGB be any angle inscribed in AMB.

Then, $\angle AGB$ is measured by $\frac{1}{2}$ arc ANB. (§ 193.)

But AC is tangent to the circle AMB. (§ 169.)

Whence, $\angle BAC$ is measured by $\frac{1}{2}$ arc ANB. (§ 197.)

Then, $\angle AGB = \angle BAC = \angle A'$.

Hence, every angle inscribed in the segment AMB is equal to A'. (§ 195.)

EXERCISES.

86. Given the middle point of a chord of a circle, to construct the chord.

87. To circumscribe a circle about a given rectangle.

88. To draw a line tangent to a given circle and parallel to a given straight line.

89. To draw a line tangent to a given circle and perpendicular to a given straight line.

90. Through a given point to draw a straight line forming an isosceles triangle with two given intersecting lines.

91. Given the base, an adjacent angle, and the altitude of a triangle, to construct the triangle.

92. Given the base, an adjacent side, and the altitude of a triangle, to construct the triangle.

93. To construct a rhombus, having given its base and altitude.

94. Given the altitude and the sides including the vertical angle of a triangle, to construct the triangle.

95. Given the altitude of a triangle, and the angles at the extremities of the base, to construct the triangle.

96. To construct an isosceles triangle, having given the base and the radius of the circumscribed circle.

97. To construct a square, having given its diagonal. (§ 196.)

98. To construct a right triangle, having given the hypotenuse and the length of the perpendicular drawn to it from the vertex of the right angle.

99. To construct a right triangle, having given the hypotenuse and a leg.

100. Given two sides of a triangle and the perpendicular to one of them from the opposite vertex, to construct the triangle.

101. To describe a circle of given radius tangent to two given intersecting lines.

102. To describe a circle tangent to a given straight line, having its centre at a given point.

103. To construct a circle having its centre in a given line, and passing through two given points without the line.

104. In a given straight line to find a point equally distant from two given straight lines.

105. Given a side and the diagonals of a parallelogram, to construct the parallelogram.

106. Through a given point within a circle to draw a chord equal to a given chord. (§ 164.)

107. Through a given point to describe a circle tangent to a given straight line at a given point.

108. Through a given point to describe a circle of given radius, tangent to a given straight line.

109. To describe a circle of given radius tangent to two given circles.

110. To describe a circle tangent to two given parallels, and passing through a given point.

111. To describe a circle of given radius, tangent to a given line and a given circle.

112. To construct a parallelogram, having given a side, an angle, and the diagonal drawn from the vertex of the angle.

113. In a given triangle to inscribe a rhombus, having one of its angles coincident with an angle of the triangle.

114. To describe a circle touching two given straight lines, one of them at a given point.

115. In a given sector to inscribe a circle.

116. In a given right triangle to inscribe a square, having one of its angles coincident with the right angle of the triangle.

117. To inscribe a square in a given rhombus.

118. To draw a common tangent to two given circles.

119. Given the base, the altitude, and the vertical angle of a triangle, to construct the triangle. (§ 226.)

120. Given the base of a triangle, its vertical angle, and the median drawn to the base, to construct the triangle.

121. To construct a triangle, having given the middle points of its sides. (§ 130.)

122. Through a vertex of a triangle to draw a straight line equally distant from the other vertices.

123. Given two sides of a triangle, and the median drawn to the third side, to construct the triangle.

124. Given the base, the altitude, and the diameter of the circumscribed circle of a triangle, to construct the triangle.

Note. For additional exercises on Book II., see p. 224.

BOOK III.

THEORY OF PROPORTION. — SIMILAR POLYGONS.

DEFINITIONS.

227. A *Proportion* is a statement that two ratios are equal.

The statement that the ratio of a to b is equal to the ratio of c to d, may be written in either of the forms

$$a : b = c : d, \text{ or } \frac{a}{b} = \frac{c}{d}.$$

228. The first and fourth terms of a proportion are called the *extremes*, and the second and third the *means*.

The first and third terms are called the *antecedents*, and the second and fourth the *consequents*.

Thus, in the proportion $a : b = c : d$, a and d are the extremes, b and c the means, a and c the antecedents, and b and d the consequents.

229. If the means of a proportion are equal, either mean is called a *mean proportional* between the first and last terms, and the last term is called a *third proportional* to the first and second terms.

Thus, in the proportion $a : b = b : c$, b is a mean proportional between a and c, and c a third proportional to a and b.

230. A *fourth proportional* to three quantities is the fourth term of a proportion, whose first three terms are the three quantities taken in their order.

Thus, in the proportion $a : b = c : d$, d is a fourth proportional to a, b, and c.

PROPOSITION I. THEOREM.

231. *In any proportion, the product of the extremes is equal to the product of the means.*

Let the proportion be $a : b = c : d$.

To prove $ad = bc$.

By § 227, $\frac{a}{b} = \frac{c}{d}$.

Multiplying both members of the equation by bd, we have

$$ad = bc.$$

232. COR. *The mean proportional between two quantities is equal to the square root of their product.*

Let the proportion be $a : b = b : c$. (1)

To prove $b = \sqrt{ac}$.

We have from (1), $b^2 = ac$. (§ 231.)

Whence, $b = \sqrt{ac}$.

PROPOSITION II. THEOREM.

233. (Converse of Prop. I.) *If the product of two quantities is equal to the product of two others, one pair may be made the extremes, and the other pair the means, of a proportion.*

Let $ad = bc$.

To prove $a : b = c : d$.

Dividing both members of the given equation by bd,

$$\frac{ad}{bd} = \frac{bc}{bd},$$

or, $\frac{a}{b} = \frac{c}{d}$.

That is, $a : b = c : d$.

In like manner, $a : c = b : d$,

$b : a = d : c$, etc.

Proposition III. Theorem.

234. *In any proportion, the terms are in proportion by* Alternation; *that is, the first term is to the third as the second term is to the fourth.*

Let the proportion be $a : b = c : d$. (1)

To prove $a : c = b : d$.

We have from (1), $ad = bc$. (§ 231.)

Whence, $a : c = b : d$. (§ 233.)

Proposition IV. Theorem.

235. *In any proportion, the terms are in proportion by* Inversion; *that is, the second term is to the first as the fourth term is to the third.*

Let the proportion be $a : b = c : d$. (1)

To prove $b : a = d : c$.

We have from (1), $ad = bc$. (§ 231.)

Whence, $b : a = d : c$. (§ 233.)

Proposition V. Theorem.

236. *In any proportion, the terms are in proportion by* Composition; *that is, the sum of the first two terms is to the first term as the sum of the last two terms is to the third term.*

Let the proportion be $a : b = c : d$. (1)

To prove, $a + b : a = c + d : c$.

We have from (1), $ad = bc$. (§ 231.)

Adding each member of the equation to ac,

$$ac + ad = ac + bc,$$

or, $$a(c + d) = c(a + b).$$

Whence, $a + b : a = c + d : c$. (§ 233.)

In like manner, $a + b : b = c + d : d$.

Proposition VI. Theorem.

237. *In any proportion, the terms are in proportion by* Division; *that is, the difference of the first two terms is to the first term as the difference of the last two terms is to the third term.*

Let the proportion be $a : b = c : d$, (1)

in which a is greater than b, and c greater than d.

To prove $a - b : a = c - d : c$.

We have from (1), $ad = bc$. (§ 231.)

Subtracting both members of the equation from ac,

$$ac - ad = ac - bc,$$

or, $$a(c - d) = c(a - b).$$

Whence, $a - b : a = c - d : c$. (§ 233.)

In like manner, $a - b : b = c - d : d$.

Proposition VII. Theorem.

238. *In any proportion, the terms are in proportion by* Composition and Division; *that is, the sum of the first two terms is to their difference as the sum of the last two terms is to their difference.*

Let the proportion be $a : b = c : d$, (1)

in which a is greater than b, and c greater than d.

To prove $a + b : a - b = c + d : c - d$.

We have from (1), $\frac{a+b}{a} = \frac{c+d}{c}$, (§ 236.)

and $\frac{a-b}{a} = \frac{c-d}{c}$. (§ 237.)

Dividing the first equation by the second, we have

$$\frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

That is, $a + b : a - b = c + d : c - d$.

Proposition VIII. Theorem.

239. *In a series of equal ratios, any antecedent is to its consequent as the sum of all the antecedents is to the sum of all the consequents.*

Let $a : b = c : d = e : f.$

To prove $a : b = a + c + e : b + d + f.$

Let r denote the value of each of the given ratios.

Then, $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = r.$

Whence, $a = br,\ c = dr,\ e = fr.$

Then, $a + c + e = br + dr + fr$
$= r(b + d + f).$

Whence, $\frac{a + c + e}{b + d + f} = r = \frac{a}{b}.$

That is, $a : b = a + c + e : b + d + f.$

Proposition IX. Theorem.

240. *Equimultiples of two quantities are in the same ratio as the quantities themselves.*

Let a and b be any two quantities.

To prove $ma : mb = a : b.$

We have, $\frac{ma}{mb} = \frac{a}{b}.$

Whence, $ma : mb = a : b.$

Proposition X. Theorem.

241. *In any number of proportions, the products of the corresponding terms are in proportion.*

Let, $a : b = c : d,$

and $e : f = g : h.$

To prove $ae : bf = cg : dh.$

We have, $\frac{a}{b} = \frac{c}{d}$, and $\frac{e}{f} = \frac{g}{h}$.

Multiplying these equations, we obtain

$$\frac{a}{b} \times \frac{e}{f} = \frac{c}{d} \times \frac{g}{h},$$

or, $$\frac{ae}{bf} = \frac{cg}{dh}.$$

That is, $$ae : bf = cg : dh.$$

PROPOSITION XI. THEOREM.

242. *In any proportion, like powers or like roots of the terms are in proportion.*

Let the proportion be $a : b = c : d$. (1)

To prove $a^n : b^n = c^n : d^n$.

We have from (1), $\frac{a}{b} = \frac{c}{d}$.

Raising both members to the nth power,

$$\frac{a^n}{b^n} = \frac{c^n}{d^n}.$$

That is, $$a^n : b^n = c^n : d^n.$$

In like manner, $\sqrt[n]{a} : \sqrt[n]{b} = \sqrt[n]{c} : \sqrt[n]{d}$.

NOTE. The ratio of two magnitudes of the same kind is equal to the ratio of their numerical measures when referred to a common unit (§ 183).

Hence, in any proportion involving the ratio of two magnitudes of the same kind, we may regard the ratio of the magnitudes as replaced by the ratio of their numerical measures when referred to a common unit.

Thus, let AB, CD, EF, and GH be four lines such that

$$AB : CD = EF : GH.$$

Then, $AB \times GH = CD \times EF$. (§ 231.)

This means that the product of the *numerical measures* of AB and GH is equal to the product of the *numerical measures* of CD and EF.

An interpretation of this nature must be given to all applications of §§ 231, 232, 241, and 242.

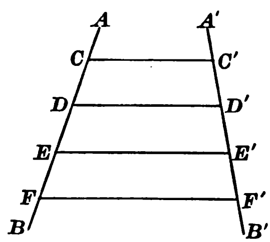
EXERCISES.

1. Find a fourth proportional to 65, 80, and 91.
2. Find a mean proportional between 28 and 63.
3. Find a third proportional to $\frac{3}{4}$ and $\frac{5}{6}$.
4. What is the second term of a proportion whose first, third, and fourth terms are 45, 160, and 224?

PROPORTIONAL LINES.

PROPOSITION XII. THEOREM.

243. *If a series of parallels, cutting two straight lines, intercept equal distances on one of these lines, they also intercept equal distances on the other.*



Let the lines AB and $A'B'$ be cut by the parallels CC', DD', EE', and FF', so that

$$CD = DE = EF.$$

To prove $C'D' = D'E' = E'F'$.

In the trapezoid $CC'E'E$, DD' is parallel to the bases, and bisects CE; it therefore bisects $C'E'$. (§ 133.)

That is, $C'D' = D'E'$. (1)

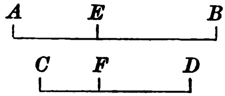
In like manner, in the trapezoid $DD'F'F$, EE' is parallel to the bases, and bisects DF.

Whence, $D'E' = E'F'$. (2)

From (1) and (2), we have

$$C'D' = D'E' = E'F'.$$

244. Def. Two straight lines are said to be *divided proportionally* when their corresponding segments are in the same ratio as the lines themselves.



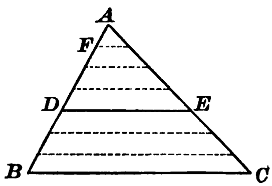
Thus, the lines AB and CD are divided proportionally if

$$\frac{AE}{CF}=\frac{BE}{DF}=\frac{AB}{CD}.$$

Proposition XIII. Theorem.

245. *A parallel to one side of a triangle divides the other two sides proportionally.*

Case I. *When the segments of each side are commensurable.*



Let DE be parallel to the side BC of the triangle ABC.

Let AD and DB be commensurable.

To prove $$\frac{AD}{DB}=\frac{AE}{EC}.$$

Let AF be a common measure of AD and DB; and suppose it to be contained 4 times in AD, and 3 times in DB.

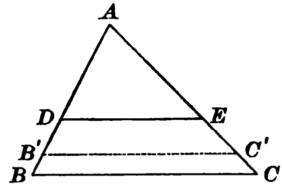
Then, $$\frac{AD}{DB}=\frac{4}{3}. \qquad (1)$$

Drawing parallels to BC through the several points of division, AE will be divided into 4 parts, and EC into 3 parts, all of which parts will be equal. (§ 243.)

Whence, $$\frac{AE}{EC}=\frac{4}{3}. \qquad (2)$$

From (1) and (2), $$\frac{AD}{DB}=\frac{AE}{EC}. \qquad \text{(Ax. 1.)}$$

CASE II. *When the segments of each side are incommensurable.*



Let DE be parallel to the side BC of the triangle ABC.

Let AD and DB be incommensurable.

To prove $$\frac{AD}{DB} = \frac{AE}{EC}.$$

Let AD be divided into any number of equal parts, and let one of these parts be applied to DB as a measure.

Since AD and DB are incommensurable, a certain number of the parts will extend from D to B', leaving a remainder BB' less than one of the parts.

Draw $B'C'$ parallel to BC.

Then, $$\frac{AD}{DB'} = \frac{AE}{EC'}. \qquad \text{(§ 245, Case I.)}$$

Now let the number of subdivisions of AD be indefinitely increased.

Then the length of each part will be indefinitely diminished, and the remainder BB' will approach the limit 0.

Then, $\frac{AD}{DB'}$ will approach the limit $\frac{AD}{DB}$,

and $\frac{AE}{EC'}$ will approach the limit $\frac{AE}{EC}$.

By the Theorem of Limits, these limits are equal. (§ 188.)

Whence, $$\frac{AD}{DB} = \frac{AE}{EC}.$$

246. COR. The result of § 245 may be written

$$AD : DB = AE : EC. \qquad (1)$$

Then, $$AD + DB : AD = AE + EC : AE. \qquad \text{(§ 236.)}$$

That is, $AB : AD = AC : AE.$ (2)

In like manner, $AB : DB = AC : EC.$ (3)

From (2), $AB : AC = AD : AE.$ (§ 234.)

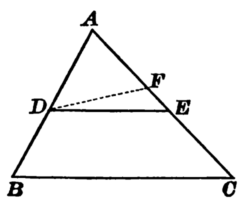
Also, from (3), $AB : AC = DB : EC.$

Therefore, $$\frac{AB}{AC} = \frac{AD}{AE} = \frac{DB}{EC}.$$ (4)

247. Sch. The proportions (1), (2), (3), and (4), of § 246, are all included in the general statement of § 245.

Proposition XIV. Theorem.

248. (Converse of Prop. XIII.) *A line which divides two sides of a triangle proportionally is parallel to the third side.*



In the triangle ABC, let DE be drawn so that
$$\frac{AB}{AD} = \frac{AC}{AE}.$$

To prove DE parallel to BC.

Let DF be drawn parallel to BC.

Then, $$\frac{AB}{AD} = \frac{AC}{AF}.$$ (§ 245.)

But by hypothesis, $$\frac{AB}{AD} = \frac{AC}{AE}.$$

Then, $$\frac{AC}{AE} = \frac{AC}{AF}.$$

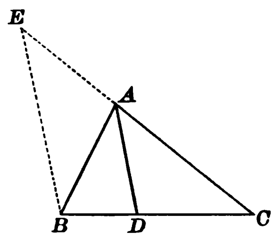
Whence, $AE = AF.$

Then DF must coincide with DE. (Ax. 5.)

Therefore, DE is parallel to BC.

Proposition XV. Theorem.

249. *In any triangle, the bisector of an angle divides the opposite side into segments proportional to the adjacent sides.*



Let AD be the bisector of the angle A of the triangle ABC, meeting the side BC at D.

To prove $$\frac{DB}{DC}=\frac{AB}{AC}.$$

Draw BE parallel to AD, meeting CA produced at E.

Then since the parallels AD and BE are cut by AB,

$$\angle ABE = \angle BAD. \qquad (§ 72.)$$

And since the parallels AD and BE are cut by CE,

$$\angle AEB = \angle CAD. \qquad (§ 74.)$$

But by hypothesis,

$$\angle BAD = \angle CAD.$$

Whence, $$\angle ABE = \angle AEB.$$

Therefore, $$AB = AE. \qquad (§ 91.)$$

Now since AD is parallel to the side BE of the triangle BCE,

$$\frac{DB}{DC}=\frac{AE}{AC}. \qquad (§ 245.)$$

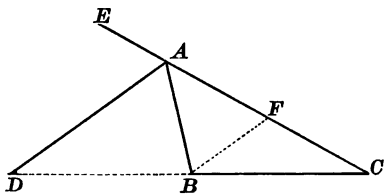
Putting for AE its equal AB, we have

$$\frac{DB}{DC}=\frac{AB}{AC}.$$

Proposition XVI. Theorem.

250. *In any triangle, the bisector of an exterior angle divides the opposite side externally into segments proportional to the adjacent sides.*

Note. The theorem does not hold for *isosceles* triangles.



Let AD be the bisector of the exterior angle BAE of the triangle ABC, meeting CB produced at D.

To prove $$\frac{DB}{DC} = \frac{AB}{AC}.$$

Draw BF parallel to AD, meeting AC at F.

Then since the parallels AD and BF are cut by AB,

$$\angle ABF = \angle BAD. \qquad (\S\ 72.)$$

And since the parallels AD and BF are cut by CE,

$$\angle AFB = \angle EAD. \qquad (\S\ 74.)$$

But by hypothesis,

$$\angle BAD = \angle EAD.$$

Whence, $\angle ABF = \angle AFB.$

Therefore, $AB = AF. \qquad (\S\ 91.)$

Now since BF is parallel to the side AD of the triangle ACD,

$$\frac{DB}{DC} = \frac{AF}{AC}. \qquad (\S\ 245.)$$

Putting for AF its equal AB, we have

$$\frac{DB}{DC} = \frac{AB}{AC}.$$

251. SCH. The *segments* of a limited straight line by a point are understood to mean the distances from the point to the extremities of the line, whether the point is in the line itself, or in the line produced.

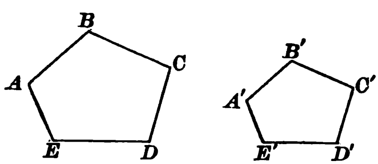
EXERCISES.

5. The sides of a triangle are $AB = 8$, $BC = 6$, and $CA = 7$; find the segments into which BC is divided by the bisector of the angle A.

6. The sides of a triangle are $AB = 5$, $BC = 7$, and $CA = 8$; find the segments into which BC is divided by the bisector of the exterior angle at A.

SIMILAR POLYGONS.

252. DEF. Two polygons are said to be *similar* when they are mutually equiangular (§ 122), and have their homologous sides proportional (§ 123).



Thus, the polygons $ABCDE$ and $A'B'C'D'E'$ are similar if

$$\angle A = \angle A', \ \angle B = \angle B', \ \angle C = \angle C', \text{ etc.,}$$

and,
$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}, \text{ etc.}$$

253. SCH. I. The following forms of the definition of § 252 are given for convenience of reference:

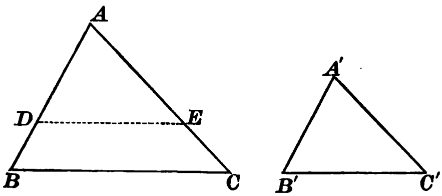
I. *In similar polygons, the homologous angles are equal.*

II. *In similar polygons, the homologous sides are proportional.*

254. SCH. II. *In similar triangles, the homologous sides lie opposite the equal angles.*

Proposition XVII. Theorem.

255. *Two triangles are similar when they are mutually equiangular.*



In the triangles ABC and $A'B'C'$, let

$$\angle A = \angle A', \ \angle B = \angle B', \text{ and } \angle C = \angle C'.$$

To prove ABC and $A'B'C'$ similar.

Superpose the triangle $A'B'C'$ upon ABC, so that $\angle A'$ shall coincide with $\angle A$; the side $B'C'$ falling at DE.

Since $\angle ADE = \angle B$, DE is parallel to BC. (§ 76.)

Therefore, $$\frac{AB}{AD} = \frac{AC}{AE}.$$ (§ 245.)

That is, $$\frac{AB}{A'B'} = \frac{AC}{A'C'}. \quad (1)$$

In like manner, by superposing $A'B'C'$ upon ABC so that $\angle B'$ shall coincide with $\angle B$, we may prove

$$\frac{AB}{A'B'} = \frac{BC}{B'C'}. \quad (2)$$

From (1) and (2), $$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}.$$

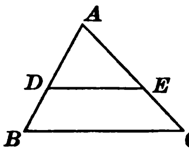
Then ABC and $A'B'C'$ have their homologous sides proportional, and are similar. (§ 252.)

256. Cor. I. *Two triangles are similar when two angles of one are equal respectively to two angles of the other.*

For their remaining angles are equal each to each. (§ 86.)

257. Cor. II. *Two right triangles are similar when an acute angle of one is equal to an acute angle of the other.*

258. COR. III. *If a line be drawn between two sides of a triangle parallel to the third side, the triangle formed is similar to the given triangle.*



Let DE be parallel to the side BC of the triangle ABC.

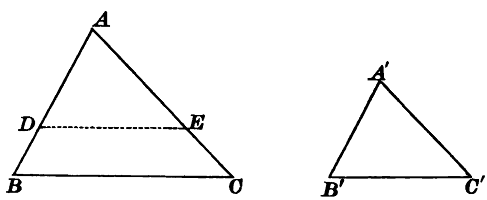
To prove the triangle ADE similar to ABC.

We have, $\angle ADE = \angle B$. (§ 74.)

Whence, the triangle ADE is similar to ABC. (§ 256.)

PROPOSITION XVIII. THEOREM.

259. *Two triangles are similar when their homologous sides are proportional.*



In the triangles ABC and $A'B'C'$, let

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}.$$

To prove ABC and $A'B'C'$ similar.

Take $AD = A'B'$, and $AE = A'C'$, and draw DE.

Then from the given proportion, we have

$$\frac{AB}{AD} = \frac{AC}{AE}.$$

Whence, DE is parallel to BC. (§ 248.)

Then the triangles ABC and ADE are similar. (§ 258.)

Therefore, $\frac{AB}{AD} = \frac{BC}{DE}$, or $\frac{AB}{A'B'} = \frac{BC}{DE}$. (§ 253, II.)

But by hypothesis, $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'}$.

Whence, $DE = B'C'$.

Therefore, $\triangle ADE = \triangle A'B'C'$. (§ 69.)

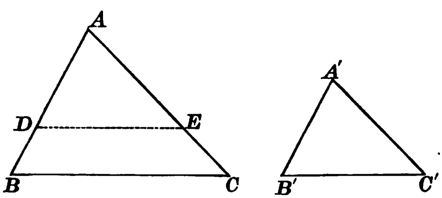
But ADE has been proved similar to ABC.

Hence, $A'B'C'$ is similar to ABC.

Note. To prove that two polygons in general are similar, it must be shown that they are mutually equiangular, and have their homologous sides proportional (§ 252); but in the case of two *triangles*, each of these conditions involves the other (§§ 255, 259), so that it is only necessary to show that one of the tests of similarity is satisfied.

Proposition XIX. Theorem.

260. *Two triangles are similar when they have an angle of one equal to an angle of the other, and the sides including these angles proportional.*



In the triangles ABC and $A'B'C'$, let

$$\angle A = \angle A', \text{ and } \frac{AB}{A'B'} = \frac{AC}{A'C'}.$$

To prove ABC and $A'B'C'$ similar.

Superpose the triangle $A'B'C'$ upon ABC, so that $\angle A'$ shall coincide with $\angle A$; the side $B'C'$ falling at DE.

Then by hypothesis, $\dfrac{AB}{AD} = \dfrac{AC}{AE}$.

Whence, DE is parallel to BC. (§ 248.)

Then the triangles ABC and ADE are similar. (§ 258.)

That is, the triangles ABC and $A'B'C'$ are similar.

Proposition XX. Theorem.

261. *Two triangles are similar when their sides are parallel each to each, or perpendicular each to each.*

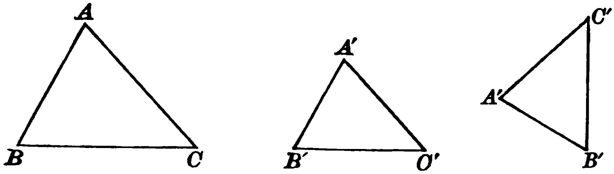


Fig. 1. *Fig. 2.* *Fig. 3.*

Let the sides AB, AC, and BC of the triangle ABC be parallel respectively to the sides $A'B'$, $A'C'$, and $B'C'$ of the triangle $A'B'C'$ (Fig. 2), and perpendicular respectively to the sides $A'B'$, $A'C'$, and $B'C'$ of the triangle $A'B'C'$ (Fig. 3).

To prove the triangles similar.

Since the sides of the angles A and A' are parallel each to each, or perpendicular each to each, the angles are either equal or supplementary. (§§ 79, 80, 81.)

In like manner, B and B', and C and C', are either equal or supplementary.

We may then make five hypotheses with regard to the angles of the triangles, R denoting a right angle:

1. $A + A' = 2\,R$, $B + B' = 2\,R$, $C + C' = 2\,R$.
2. $A + A' = 2\,R$, $B + B' = 2\,R$, $C = C'$.
3. $A + A' = 2\,R$, $B = B'$, $C + C' = 2\,R$.
4. $A = A'$, $B + B' = 2\,R$, $C + C' = 2\,R$.
5. $A = A'$, and $B = B'$; then, $C = C'$. (§ 86.)

But the first four hypotheses are impossible; for, in either case, the sum of the angles of the two triangles would exceed four right angles. (§ 82.)

We can then have only $A = A'$, $B = B'$, and $C = C'$.

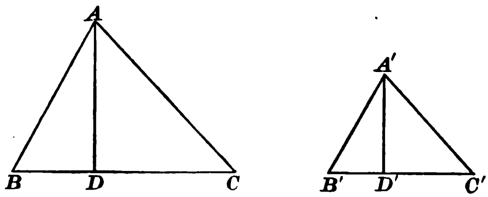
Therefore, the triangles are similar. (§ 255.)

262. Sch. 1. *In similar triangles whose sides are parallel each to each, the parallel sides are homologous.*

2. *In similar triangles whose sides are perpendicular each to each, the perpendicular sides are homologous.*

Proposition XXI. Theorem.

263. *The homologous altitudes of two similar triangles are in the same ratio as any two homologous sides.*



Let AD and $A'D'$ be homologous altitudes of the similar triangles ABC and $A'B'C'$.

To prove $$\frac{AD}{A'D'}=\frac{AB}{A'B'}=\frac{AC}{A'C'}=\frac{BC}{B'C'}.$$

Since the triangles ABC and $A'B'C'$ are similar,

$$\angle B=\angle B'. \qquad (\S\ 253, \text{I.})$$

Then the right triangles ABD and $A'B'D'$ are similar. (§ 257.)

Whence, $$\frac{AD}{A'D'}=\frac{AB}{A'B'}. \qquad (\S\ 253, \text{II.})$$

But since the triangles ABC and $A'B'C'$ are similar,

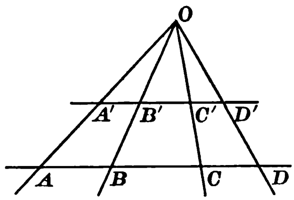
$$\frac{AB}{A'B'}=\frac{AC}{A'C'}=\frac{BC}{B'C'}. \qquad (\S\ 253, \text{II.})$$

Whence, $$\frac{AD}{A'D'}=\frac{AB}{A'B'}=\frac{AC}{A'C'}=\frac{BC}{B'C'}.$$

264. Sch. In two similar triangles, any two homologous lines are in the same ratio as any two homologous sides.

Proposition XXII. Theorem.

265. *If two parallels are cut by three or more straight lines passing through a common point, the corresponding segments are proportional.*



Let the parallels AD and $A'D'$ be cut by the lines OA, OB, OC, and OD, in the points A, B, C, D, and A', B', C', D', respectively.

To prove $$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}.$$

The triangles OAB and $OA'B'$ are similar. (§ 258.)

Whence, $$\frac{AB}{A'B'} = \frac{OB}{OB'}. \quad (1)$$

In like manner, the triangles OBC and $OB'C'$ are similar.

Whence, $$\frac{OB}{OB'} = \frac{BC}{B'C'} = \frac{OC}{OC'}. \quad (2)$$

Again, the triangles OCD and $OC'D'$ are similar.

Whence, $$\frac{OC}{OC'} = \frac{CD}{C'D'}. \quad (3)$$

From (1), (2), and (3), we have

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}.$$

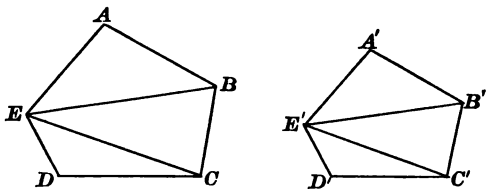
EXERCISES.

7. The sides of a triangle are 5, 7, and 9. The shortest side of a similar triangle is 14. What are the other two sides ?

8. Two isosceles triangles are similar when their vertical angles are equal.

PROPOSITION XXIII. THEOREM.

266. *Two polygons are similar when they are composed of the same number of triangles, similar each to each, and similarly placed.*



In the polygons $A - E$ and $A' - E'$, let the triangle ABE be similar to $A'B'E'$, BCE to $B'C'E'$, and CDE to $C'D'E'$.

To prove $A - E$ and $A' - E'$ similar.

Since the triangles ABE and $A'B'E'$ are similar,

$$\angle A = \angle A'. \qquad (\S\ 253,\ \text{I.})$$

Also,

$$\angle ABE = \angle A'B'E'. \qquad (1)$$

And since the triangles BCE and $B'C'E'$ are similar,

$$\angle EBC = \angle E'B'C'. \qquad (2)$$

Adding (1) and (2),

$$\angle ABC = \angle A'B'C'.$$

In like manner, $\angle BCD = \angle B'C'D'$, etc.

That is, $A - E$ and $A' - E'$ are mutually equiangular.

Again, since ABE is similar to $A'B'E'$, and BCE to $B'C'E'$,

$$\frac{AB}{A'B'} = \frac{BE}{B'E'}, \text{ and } \frac{BE}{B'E'} = \frac{BC}{B'C'}. \qquad (\S\ 253,\ \text{II.})$$

Therefore,

$$\frac{AB}{A'B'} = \frac{BC}{B'C'}.$$

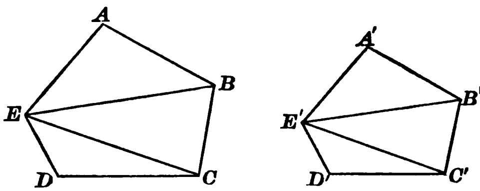
In like manner, $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{CD}{C'D'}$, etc.

That is, $A - E$ and $A' - E'$ have their homologous sides proportional.

Therefore, $A - E$ and $A' - E'$ are similar. (§ 252.)

Proposition XXIV. Theorem.

267. (Converse of Prop. XXIII.) *Two similar polygons may be decomposed into the same number of triangles, similar each to each, and similarly placed.*



Let E and E' be homologous vertices of the similar polygons $A-E$ and $A'-E'$, and draw $EB, EC, E'B'$, and $E'C'$.

To prove the triangle ABE similar to $A'B'E'$, BCE to $B'C'E'$, and CDE to $C'D'E'$.

Since the polygons are similar,

$$\angle A = \angle A', \text{ and } \frac{AE}{A'E'} = \frac{AB}{A'B'}. \qquad (§\ 253.)$$

Then, the triangles ABE and $A'B'E'$ are similar. (§ 260.)

Again, since the polygons are similar,

$$\angle ABC = \angle A'B'C'. \qquad (1)$$

And since the triangles ABE and $A'B'E'$ are similar,

$$\angle ABE = \angle A'B'E'. \qquad (2)$$

Subtracting (2) from (1), we have

$$\angle EBC = \angle E'B'C'.$$

Also, since the polygons are similar,

$$\frac{AB}{A'B'} = \frac{BC}{B'C'}.$$

And since the triangles ABE and $A'B'E'$ are similar,

$$\frac{AB}{A'B'} = \frac{BE}{B'E'}.$$

Whence,

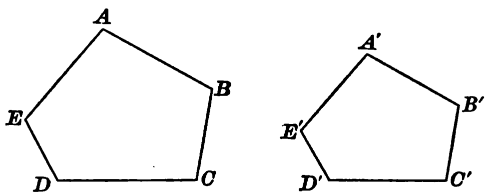
$$\frac{BC}{B'C'} = \frac{BE}{B'E'}.$$

Then, the triangles BCE and $B'C'E'$ are similar. (§ 260.)

In like manner, we may prove the triangles CDE and $C'D'E'$ similar.

PROPOSITION XXV. THEOREM.

268. *The perimeters of two similar polygons are in the same ratio as any two homologous sides.*



Let P and P' denote the perimeters of the similar polygons $A - E$ and $A' - E'$, respectively.

To prove $$\frac{P}{P'} = \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}, \text{ etc.}$$

Since the polygons are similar, we have

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}, \text{ etc.} \quad (\S\ 253, \text{II.})$$

Then, $$\frac{AB + BC + CD + \text{etc.}}{A'B' + B'C' + C'D' + \text{etc.}} = \frac{AB}{A'B'}. \quad (\S\ 239.)$$

Therefore, $$\frac{P}{P'} = \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}, \text{ etc.}$$

EXERCISES.

9. The base and altitude of a triangle are 5 ft. 10 in. and 3 ft. 6 in., respectively. If the homologous base of a similar triangle is 7 ft. 6 in., find its homologous altitude.

10. The perimeters of two similar polygons are 119 and 68; if a side of the first is 21, what is the homologous side of the second?

11. AB is the hypotenuse of a right triangle ABC. If perpendiculars be drawn to AB at A and B, meeting AC produced at D, and BC produced at E, prove the triangles ACE and BCD similar.

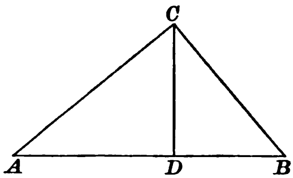
Proposition XXVI. Theorem.

269. *If a perpendicular be drawn from the vertex of the right angle to the hypotenuse of a right triangle,*

I. *The triangles formed are similar to the whole triangle, and to each other.*

II. *The perpendicular is a mean proportional between the segments of the hypotenuse.*

III. *Either leg is a mean proportional between the whole hypotenuse and the adjacent segment.*



Let CD be perpendicular to the hypotenuse AB of the right triangle ABC.

I. To prove that the triangles ACD and BCD are similar to ABC, and to each other.

In the right triangles ACD and ABC, $\angle A$ is common.
Whence, ACD is similar to ABC. (§ 257.)
In like manner, BCD is similar to ABC.
Then ACD and BCD are similar, for each is similar to ABC.

II. To prove $AD : CD = CD : BD$.

Since the triangles ACD and BCD are similar,

$\angle ACD = \angle B$, and $\angle A = \angle BCD$. (§ 253, I.)

Then AD, the side of ACD opposite $\angle ACD$, is homologous to CD, the side of BCD opposite $\angle B$; and CD, the side of ACD opposite $\angle A$, is homologous to BD, the side of BCD opposite $\angle BCD$. (§ 254.)

Whence, $AD : CD = CD : BD$. (§ 253, II.)

III. To prove $AB : AC = AC : AD.$

Since the triangles ABC and ACD are similar,

$$\angle ACB = \angle ADC, \text{ and } \angle B = \angle ACD. \quad (\S\ 253, \text{I.})$$

Then AB, the side of ABC opposite $\angle ACB$, is homologous to AC, the side of ACD opposite $\angle ADC$; and AC, the side of ABC opposite $\angle B$, is homologous to AD, the side of ACD opposite $\angle ACD$. (§ 254.)

Whence, $AB : AC = AC : AD.$ (§ 253, II.)

In like manner, $AB : BC = BC : BD.$

270. Cor. I. The three proportions of § 269 give

$$\overline{CD}^2 = AD \times BD,$$

$$\overline{AC}^2 = AB \times AD,$$

and

$$\overline{BC}^2 = AB \times BD. \quad (\S\ 231.)$$

Hence, *if a perpendicular be drawn from the vertex of the right angle to the hypotenuse of a right triangle,*

I. *The square of the perpendicular is equal to the product of the segments of the hypotenuse.*

II. *The square of either leg is equal to the product of the whole hypotenuse and the adjacent segment.*

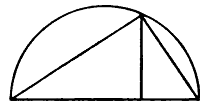
As stated in the Note, p. 125, the above equations mean simply that the square of the *numerical measure* of CD is equal to the product of the *numerical measures* of AD and BD, etc.

271. Cor. II. Dividing the second equation of § 270 by the third, we obtain

$$\frac{\overline{AC}^2}{\overline{BC}^2} = \frac{AB \times AD}{AB \times BD} = \frac{AD}{BD}.$$

That is, *if a perpendicular be drawn from the vertex of the right angle to the hypotenuse of a right triangle, the squares of the legs are proportional to the adjacent segments of the hypotenuse.*

272. Cor. III. Since an angle inscribed in a semicircle is a right angle (§ 196), it follows that:



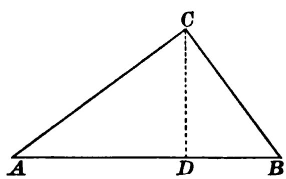
If a perpendicular be drawn from any point in the circumference of a circle to a diameter,

I. *The perpendicular is a mean proportional between the segments of the diameter.*

II. *The chord joining the point to either extremity of the diameter is a mean proportional between the whole diameter and the adjacent segment.*

Proposition XXVII. Theorem.

273. *In any right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs.*



Let AB be the hypotenuse of the right triangle ABC.

To prove $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$.

Draw CD perpendicular to AB.

Then, $\overline{AC}^2 = AB \times AD,$

and $\overline{BC}^2 = AB \times BD.$ (§ 270, II.)

Adding, $\overline{AC}^2 + \overline{BC}^2 = AB \times (AD + BD)$

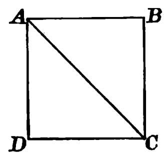
$= AB \times AB = \overline{AB}^2.$

274. Cor. I. It follows from § 273 that

$$\overline{AC}^2 = \overline{AB}^2 - \overline{BC}^2, \text{ and } \overline{BC}^2 = \overline{AB}^2 - \overline{AC}^2.$$

That is, *in any right triangle, the square of either leg is equal to the square of the hypotenuse, minus the square of the other leg.*

275. Cor. II. If AC is a diagonal of the square $ABCD$, we have



$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 = \overline{AB}^2 + \overline{AB}^2 = 2\overline{AB}^2.$$

Dividing each member of the equation by $\overline{AB}^2$,

$$\frac{\overline{AC}^2}{\overline{AB}^2} = 2, \text{ or } \frac{AC}{AB} = \sqrt{2}.$$

Hence, *the diagonal of a square is incommensurable with its side* (§ 181).

EXERCISES.

12. What is the length of the tangent to a circle whose diameter is 16, from a point whose distance from the centre is 17?

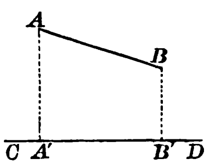
13. What is the length of the longest straight line which can be drawn on a floor 33 ft. 4 in. long, and 16 ft. 3 in. wide?

14. A ladder 32 ft. 6 in. long is placed so that it just reaches a window 26 ft. above the street; and when turned about its foot, just reaches a window 16 ft. 6 in. above the street on the other side. Find the width of the street.

15. The side of an equilateral triangle is 5; what is its altitude?

16. Find the length of the diagonal of a square whose side is 1 ft. 3 in.

276. Definitions. The *projection of a point* upon a straight line of indefinite length, is the foot of the perpendicular let fall from the point to the line.



Thus, if AA' be drawn perpendicular to CD, the projection of the point A upon the line CD is the point A'.

The *projection of a finite straight line* upon a straight line of indefinite length, is that portion of the second line included between the projections of the extremities of the first.

Thus, if AA' and BB' be drawn perpendicular to CD, the projection of the line AB upon the line CD is $A'B'$.

Proposition XXVIII. Theorem.

277. *In any triangle, the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides, minus twice the product of one of these sides and the projection of the other side upon it.*

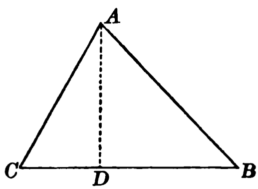


Fig. 1.

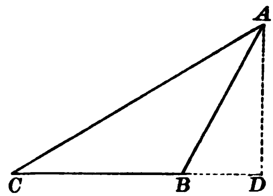


Fig. 2.

Let C be an acute angle of the triangle ABC, and draw AD perpendicular to CB, produced if necessary.

Then CD is the projection of the side AC upon BC (§ 276).

To prove $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2\,BC \times CD.$

There will be two cases according as D falls upon CB (Fig. 1), or upon CB produced (Fig. 2).

In Fig. 1, $\quad BD = BC - CD.$

In Fig. 2, $\quad BD = CD - BC.$

Squaring these equations, we have in either case

$$\overline{BD}^2 = \overline{BC}^2 + \overline{CD}^2 - 2\,BC \times CD.$$

Adding $\overline{AD}^2$ to both members,

$$\overline{AD}^2 + \overline{BD}^2 = \overline{BC}^2 + \overline{AD}^2 + \overline{CD}^2 - 2\,BC \times CD.$$

But in the right triangles ABD and ACD,

$$\overline{AD}^2 + \overline{BD}^2 = \overline{AB}^2,$$

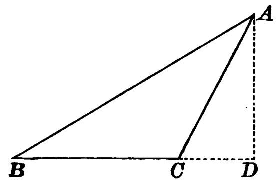
and $$\overline{AD}^2 + \overline{CD}^2 = \overline{AC}^2. \qquad (§ 273.)$$

Whence, $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2\,BC \times CD.$

Ex. 17. The square of the side of an equilateral triangle is equal to four-thirds the square of the altitude.

Proposition XXIX. Theorem.

278. *In any triangle having an obtuse angle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides, plus twice the product of one of these sides and the projection of the other side upon it.*



Let C be an obtuse angle of the triangle ABC, and draw AD perpendicular to BC produced.

Then CD is the projection of the side AC upon BC.

To prove $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 + 2\,BC \times CD$.

We have, $\qquad BD = BC + CD.$

Squaring this equation,

$$\overline{BD}^2 = \overline{BC}^2 + \overline{CD}^2 + 2\,BC \times CD.$$

Adding $\overline{AD}^2$ to both members,

$$\overline{AD}^2 + \overline{BD}^2 = \overline{BC}^2 + \overline{AD}^2 + \overline{CD}^2 + 2\,BC \times CD.$$

But in the right triangles ABD and ACD,

$$\overline{AD}^2 + \overline{BD}^2 = \overline{AB}^2,$$

and $$\overline{AD}^2 + \overline{CD}^2 = \overline{AC}^2. \qquad (\S\ 273.)$$

Whence, $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 + 2\,BC \times CD.$

EXERCISES.

18. The length of the tangent to a circle, whose diameter is 80, from a point without the circumference, is 42. What is the distance of the point from the centre?

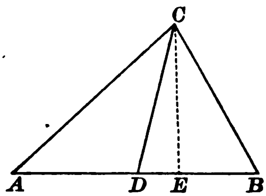
19. Find the length of the common tangent to two circles whose radii are 11 and 18, if the distance between their centres is 25.

PROPOSITION XXX. THEOREM.

279. *In any triangle, if a median be drawn from the vertex to the base,*

I. *The sum of the squares of the other two sides is equal to twice the square of half the base, plus twice the square of the median.*

II. *The difference of the squares of the other two sides is equal to twice the product of the base and the projection of the median upon the base.*



Let DE be the projection of the median CD upon the base AB of the triangle ABC; and let AC be greater than BC.

To prove I. $\overline{AC}^2 + \overline{BC}^2 = 2\,\overline{AD}^2 + 2\,\overline{CD}^2$.

II. $\overline{AC}^2 - \overline{BC}^2 = 2\,AB \times DE$.

Since E falls to the right of D, $\angle\, ADC$ is obtuse, and $\angle\, BDC$ is acute.

Hence, in the triangles ADC and BDC,

$$\overline{AC}^2 = \overline{AD}^2 + \overline{CD}^2 + 2\,AD \times DE, \qquad (\S\ 278.)$$

and $$\overline{BC}^2 = \overline{BD}^2 + \overline{CD}^2 - 2\,BD \times DE. \qquad (\S\ 277.)$$

Or since $BD = AD$,

$$\overline{AC}^2 = \overline{AD}^2 + \overline{CD}^2 + AB \times DE, \qquad (1)$$

and $$\overline{BC}^2 = \overline{AD}^2 + \overline{CD}^2 - AB \times DE. \qquad (2)$$

Adding (1) and (2), we have

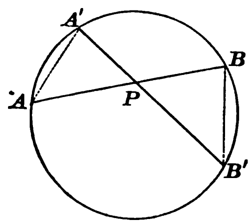
$$\overline{AC}^2 + \overline{BC}^2 = 2\,\overline{AD}^2 + 2\,\overline{CD}^2.$$

Subtracting (2) from (1),

$$\overline{AC}^2 - \overline{BC}^2 = 2\,AB \times DE.$$

PROPOSITION XXXI. THEOREM.

280. *If any two chords be drawn through a fixed point within a circle, the product of the segments of one chord is equal to the product of the segments of the other.*



Let AB and $A'B'$ be any two chords passing through the fixed point P within the circle ABB'.

To prove $AP \times BP = A'P \times B'P$.

Draw AA' and BB'.

Then in the triangles $AA'P$ and $BB'P$,

$$\angle AA'P = \angle B'BP,$$

since each is measured by $\frac{1}{2}$ arc AB'. (§ 193.)

In like manner, $\angle A'AP = \angle BB'P$.

Then the triangles $AA'P$ and $BB'P$ are similar. (§ 256.)

Whence, $AP : A'P = B'P : BP$. (§§ 253, II., 254.)

Therefore, $AP \times BP = A'P \times B'P$. (§ 231.)

281. SCH. The proportion in § 280 may be written

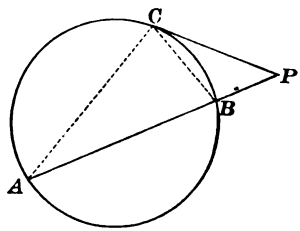
$$\frac{AP}{A'P} = \frac{B'P}{BP}, \text{ or } \frac{AP}{A'P} = \frac{1}{\frac{BP}{B'P}}.$$

If two magnitudes, such as the segments of a chord by a point, are so related that the ratio of any two values of one is equal to the *reciprocal* of the ratio of the corresponding values of the other, they are said to be *reciprocally proportional.*

Thus, *if any two chords be drawn through a fixed point within a circle, their segments are reciprocally proportional.*

Proposition XXXII. Theorem.

282. *If through a fixed point without a circle a secant and a tangent be drawn, the product of the whole secant and its external segment is equal to the square of the tangent.*



Let AP be a secant, and CP a tangent, passing through the fixed point P without the circle ABC.

To prove $\quad AP \times BP = \overline{CP}^2.$

Draw AC and BC.

Then in the triangles ACP and BCP, $\angle P$ is common.

Also, $\quad \angle A = \angle BCP,$

since each is measured by $\frac{1}{2}$ arc BC. (§§ 193, 197.)

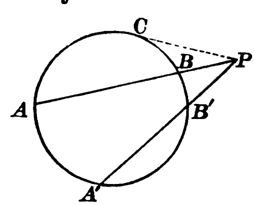
Then the triangles ACP and BCP are similar. (§ 256.)

Whence, $\quad AP : CP = CP : BP.$ (§§ 253, II., 254.)

Therefore, $\quad AP \times BP = \overline{CP}^2.$ (§ 231.)

283. Cor. I. *If through a fixed point without a circle a secant and a tangent be drawn, the tangent is a mean proportional between the whole secant and its external segment.*

284. Cor. II. Let AP and $A'P$ be any two secants of the circle ACB, intersecting without the circumference; and draw CP tangent to the circle at C.



Then, $\quad AP \times BP = \overline{CP}^2,$

and $\quad A'P \times B'P = \overline{CP}^2.$ (§ 282.)

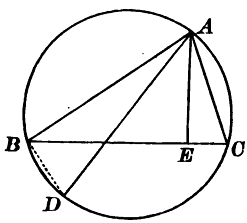
Hence, $\quad AP \times BP = A'P \times B'P.$

That is, *if any two secants be drawn through a fixed point without a circle, the product of one and its external segment is equal to the product of the other and its external segment.*

285. COR. III. *If any two secants be drawn through a fixed point without a circle, the whole secants and their external segments are reciprocally proportional* (§ 281).

PROPOSITION XXXIII. THEOREM.

286. *In any triangle, the product of any two sides is equal to the diameter of the circumscribed circle, multiplied by the perpendicular drawn to the third side from the vertex of the opposite angle.*



Let AD be the diameter of the circumscribed circle ACD of the triangle ABC, and AE the perpendicular from A to BC.

To prove $AB \times AC = AD \times AE$.

Draw BD; then, $\angle ABD$ is a right angle. (§ 196.)

Now in the right triangles ABD and ACE,

$$\angle D = \angle C,$$

since each is measured by $\frac{1}{2}$ arc AB. (§ 193.)

Then the triangles ABD and ACE are similar. (§ 257.)

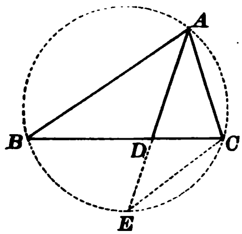
Whence, $AB : AD = AE : AC$. (§§ 253, II., 254.)

Therefore, $AB \times AC = AD \times AE$. (§ 231.)

287. COR. *In any triangle, the diameter of the circumscribed circle is equal to the product of any two sides divided by the perpendicular drawn to the third side from the vertex of the opposite angle.*

Proposition XXXIV. Theorem.

288. *In any triangle, the product of any two sides is equal to the product of the segments of the third side formed by the bisector of the opposite angle, plus the square of the bisector.*



In the triangle ABC, let AD bisect the angle A.

To prove $\quad AB \times AC = BD \times DC + \overline{AD}^2.$

Circumscribe a circle about ABC.

Produce AD to meet the circumference at E, and draw CE.

Then in the triangles ABD and ACE, by hypothesis,

$$\angle BAD = \angle CAE.$$

Also, $\quad \angle B = \angle E,$

since each is measured by $\frac{1}{2}$ arc AC. (§ 193.)

Then the triangles ABD and ACE are similar. (§ 256.)

Whence, $\quad AB : AD = AE : AC.$ (§§ 253, II., 254.)

Therefore, $\quad AB \times AC = AD \times AE$ (§ 231.)

$$= AD \times (DE + AD)$$

$$= AD \times DE + \overline{AD}^2.$$

But, $\quad AD \times DE = BD \times DC.$ (§ 280.)

Whence, $\quad AB \times AC = BD \times DC + \overline{AD}^2.$

Ex. 20. If AD is the perpendicular from A to the side BC of the triangle ABC, prove that

$$\overline{AB}^2 - \overline{AC}^2 = \overline{BD}^2 - \overline{CD}^2.$$

EXERCISES.

21. If the length of the common chord of two intersecting circles is 16, and their radii are 10 and 17, what is the distance between their centres?

22. If one leg of a right triangle is double the other, the perpendicular from the vertex of the right angle to the hypotenuse divides it into segments which are to each other as 1 to 4.

23. If two parallels to the side BC of a triangle ABC meet the sides AB and AC in D and F, and E and G, respectively, prove

$$\frac{BD}{CE} = \frac{BF}{CG} = \frac{DF}{EG}.$$

24. C and D are the middle points of a chord AB and its subtended arc. If $AD = 12$ and $CD = 8$, what is the diameter of the circle?

25. If AD and BE are the perpendiculars from the vertices A and B of the triangle ABC to the opposite sides, prove that

$$AC : DC = BC : EC.$$

26. If D is the middle point of the side BC of the triangle ABC, right-angled at C, prove that $\overline{AB}^2 - \overline{AD}^2 = 3\,\overline{CD}^2$.

27. The diameters of two concentric circles are 14 and 50 units, respectively. Find the length of a chord of the greater circle which is tangent to the smaller.

28. The length of a tangent to a circle from a point 8 units distant from the nearest point of the circumference, is 12 units. What is the diameter of the circle?

29. The non-parallel sides AD and BC of a trapezoid $ABCD$ intersect at O. If $AB = 15$, $CD = 24$, and the altitude of the trapezoid is 8, what is the altitude of the triangle OAB?

30. If the equal sides of an isosceles right triangle are each 18 units in length, what is the length of the median drawn from the vertex of the right angle?

31. The non-parallel sides of a trapezoid are each 53 units in length, and one of the parallel sides is 56 units longer than the other. Find the altitude of the trapezoid.

32. AB is a chord of a circle, and CE is any chord drawn through the middle point C of the arc AB, cutting the chord AB at D. Prove that AC is a mean proportional between CD and CE.

33. Two secants are drawn to a circle from an outside point. If their external segments are 12 and 9, while the internal segment of the former is 8, what is the internal segment of the latter?

34. If, in a triangle ABC, $\angle C = 120°$, prove that

$$\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 + AC \times BC.$$

35. BC is the base of an isosceles triangle ABC inscribed in a circle. If a chord AD be drawn cutting BC at E, prove that

$$AD : AB = AB : AE.$$

36. Two parallel chords on opposite sides of the centre of a circle are 48 units and 14 units long, respectively, and the distance between their middle points is 31 units. What is the diameter of the circle?

37. ABC is a triangle inscribed in a circle. Another circle is drawn tangent to the first externally at C, and AC and BC are produced to meet its circumference at D and E. Prove that the triangles ABC and CDE are similar. (§ 197.)

38. ABC and $A'BC$ are triangles whose vertices A and A' lie in a parallel to their common base BC. If a parallel to BC cuts AB and AC at D and E, and $A'B$ and $A'C$ at D' and E', prove that

$$DE = D'E'.$$

39. A line parallel to the bases of a trapezoid, passing through the intersection of the diagonals, and terminating in the non-parallel sides, is bisected by the diagonals.

40. If the sides of a triangle ABC are $AB = 10$, $BC = 14$, and $CA = 16$, find the lengths of the three medians. (§ 279, I.)

41. If the sides of a triangle are $AB = 4$, $AC = 5$, and $BC = 6$, find the length of the bisector of the angle A. (§§ 249, 288.)

42. The tangents to two intersecting circles from any point in their common chord produced are equal. (§ 282.)

43. If two circles intersect, their common chord produced bisects their common tangents. (§ 174.)

44. AB and AC are the two tangents to a circle from the point A. If CD be drawn perpendicular to the radius OB at D, prove

$$AB : OB = BD : CD.$$

45. ABC is a triangle inscribed in a circle. A straight line AD is drawn from A to any point of BC, and a chord BE is drawn, making $\angle ABE = \angle ADC$. Prove that

$$AB \times AC = AD \times AE.$$

46. The radius of a circle is $22\frac{1}{2}$ units. Find the length of a chord which joins the points of contact of two tangents, each 30 units in length, drawn to the circle from a point without the circumference.

47. If, in a right triangle ABC, the acute angle B is double the acute angle A, prove that $\overline{AC}^2 = 3\,\overline{BC}^2$.

48. What is the product of the segments of any chord drawn through a point 9 units from the centre of a circle whose diameter is 24 units ?

49. The hypotenuse of a right triangle is 5, and the perpendicular to it from the opposite vertex is $2\frac{2}{5}$. Find the other two sides of the triangle, and the segments into which the perpendicular divides the hypotenuse. (§ 270.)

50. State and prove the converse of Prop. XV.

51. State and prove the converse of Prop. XVI.

52. If D is the middle point of the hypotenuse AB of the right triangle ABC, prove that

$$\overline{CD}^2 = \tfrac{1}{8}\left(\overline{AB}^2 + \overline{BC}^2 + \overline{CA}^2\right).$$

53. If a line be drawn from the vertex C of an isosceles triangle ABC, meeting the base AB produced at D, prove that

$$\overline{CD}^2 - \overline{CB}^2 = AD \times BD.$$

54. If AC is the base of the isosceles triangle ABC, and AD be drawn perpendicular to BC, prove that

$$3\,\overline{AD}^2 + 2\,\overline{BD}^2 + \overline{CD}^2 = \overline{AB}^2 + \overline{BC}^2 + \overline{CA}^2.$$

55. The middle points of two chords are distant 5 and 9 units, respectively, from the middle points of their subtended arcs. If the length of the first chord is 20 units, what is the length of the second ?

56. The sides AB and BC of a triangle ABC are 9 and 16, respectively, and the length of the median drawn from A is 11. Find the side AC.

57. The diameter which bisects a certain chord whose length is $33\frac{3}{5}$ units, is 35 units in length. Find the distance from either extremity of the chord to the extremities of the diameter.

58. The equal angles of an isosceles triangle are each 30°, and the equal sides are each 8 units in length. What is the length of the base ?

59. The diagonals AC and BD of a trapezoid intersect at E. If $AE = 9$, $EC = 3$, and $BD = 16$, find BE and ED.

60. Prove the theorem of § 284 by drawing $A'B$ and AB'.

61. The parallel sides of a circumscribed trapezoid are 6 and 18, respectively, and the other two sides are equal to each other. Find the radius of the circle.

62. If two adjacent sides and one of the diagonals of a parallelogram are 7, 9, and 8, respectively, find the other diagonal.

63. An angle of a triangle is acute, right, or obtuse according as the square of the opposite side is less than, equal to, or greater than, the sum of the squares of the other two sides.

64. Is the greatest angle of a triangle whose sides are 3, 5, and 6, acute, right, or obtuse?

65. Is the greatest angle of a triangle whose sides are 8, 9, and 12, acute, right, or obtuse?

66. Is the greatest angle of a triangle whose sides are 12, 35, and 37, acute, right, or obtuse?

67. If D is the intersection of the perpendiculars from the vertices of the triangle ABC to the opposite sides, prove that

$$\overline{AB}^2 - \overline{AC}^2 = \overline{BD}^2 - \overline{CD}^2.$$

68. If a parallel to the hyoptenuse AB of the right triangle ABC meets AC and BC at D and E, prove that

$$\overline{AE}^2 + \overline{BD}^2 = \overline{AB}^2 + \overline{DE}^2.$$

69. The diameters of two circles are 12 units and 28 units, respectively, and the distance between their centres is 29 units. Find the length of the common tangent which cuts the straight line joining the centres.

70. State and prove the converse of Prop. XXVI., II.

71. The sum of the squares of the distances of any point in the circumference of a circle from the vertices of an inscribed square, is equal to twice the square of the diameter of the circle. (§ 196.)

72. The sides AB, BC, and CA, of a triangle ABC, are 169, 182, and 195 units, respectively. Find the segments into which AB and BC are divided by perpendiculars drawn from C and A, respectively. (§ 277.)

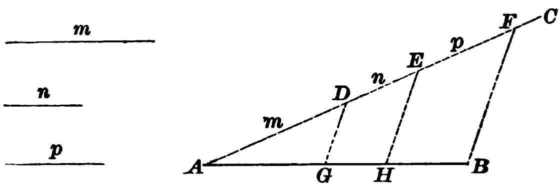
73. In a right triangle ABC is inscribed a square $DEFG$, having its vertices D and G in the hypotenuse BC, and its vertices E and F in the sides AB and AC. Prove that DE is a mean proportional between BD and CG.

NOTE. For additional exercises on Book III., see p. 224.

PROBLEMS IN CONSTRUCTION.

PROPOSITION XXXV. PROBLEM.

289. *To divide a given straight line into parts proportional to any number of given lines.*



Let AB be the given straight line.

To divide AB into parts proportional to the given lines m, n, and p.

Draw the indefinite straight line AC.

On AC take $AD = m$, $DE = n$, and $EF = p$, and draw BF.

Draw DG and EH parallel to BF, meeting AB at G and H, respectively.

Then AB is divided by the points G and H into parts proportional to m, n, and p.

For since DG is parallel to the side EH of the triangle AEH,

$$\frac{AH}{AE} = \frac{AG}{AD} = \frac{GH}{DE}. \qquad (\S\ 245.)$$

That is,

$$\frac{AH}{AE} = \frac{AG}{m} = \frac{GH}{n}. \qquad (1)$$

And since EH is parallel to the side BF of the triangle ABF,

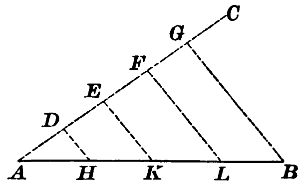
$$\frac{AH}{AE} = \frac{HB}{EF} = \frac{HB}{p}. \qquad (2)$$

From (1) and (2),

$$\frac{AG}{m} = \frac{GH}{n} = \frac{HB}{p}.$$

PROPOSITION XXXVI. PROBLEM.

290. *To divide a given straight line into any number of equal parts.*



Let AB be the given straight line.

To divide AB into four equal parts.

Draw the indefinite straight line AC.

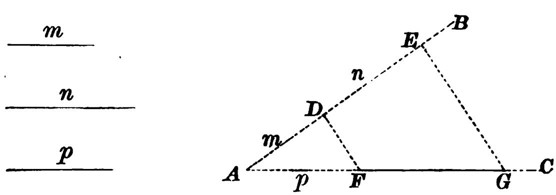
On AC take any convenient length AD, and lay off DE, EF, and FG each equal to AD.

Draw BG, and draw DH, EK, and FL parallel to BG.

Then, $AH = HK = KL = LB$. (§ 243.)

PROPOSITION XXXVII. PROBLEM.

291. *To construct a fourth proportional* (§ 230) *to three given straight lines.*



Let m, n, and p be the given straight lines.

To construct a fourth proportional to m, n, and p.

Draw the indefinite straight lines AB and AC, making any convenient angle with each other.

On AB take $AD = m$ and $DE = n$, and on AC take $AF = p$.

Draw DF, and draw EG parallel to DF.

Then FG is a fourth proportional to m, n, and p.

For since DF is parallel to the side EG of the triangle AEG,

$$AD : DE = AF : FG. \qquad (\S\ 245.)$$

That is,

$$m : n = p : FG.$$

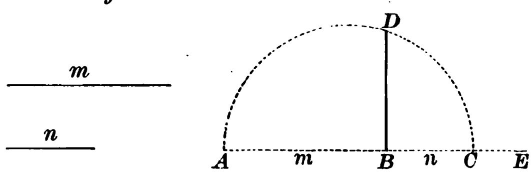
292. Cor. If AF be taken equal to n, the above proportion becomes

$$m : n = n : FG.$$

In this case, FG is a *third proportional* (§ 229) to m and n.

Proposition XXXVIII. Problem.

293. *To construct a mean proportional* (§ 229) *between two given straight lines.*



Let m and n be the given straight lines.

To construct a mean proportional between m and n.

On the indefinite straight line AE, take $AB = m$ and $BC = n$.

On AC as a diameter describe the semi-circumference ADC, and draw BD perpendicular to AC.

Then BD is a mean proportional between m and n.

For,

$$AB : BD = BD : BC. \qquad (\S\ 272, \text{I.})$$

That is,

$$m : BD = BD : n.$$

294. Sch. By aid of § 293, a line may be constructed equal to $\sqrt{a}$, where a is any number whatever.

Thus, to construct a line equal to $\sqrt{3}$, we take AB equal to 3 units, and BC equal to 1 unit.

Then, $BD = \sqrt{AB \times BC}$ (§ 231) $= \sqrt{3 \times 1} = \sqrt{3}$.

295. Def. A straight line is said to be divided by a given point in *extreme and mean ratio* when one of the segments (§ 251) is a mean proportional between the whole line and the other segment.

Thus, the line AB is divided *internally* in extreme and mean ratio at the point C if

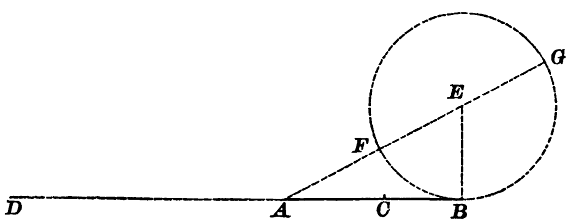
$$AB : AC = AC : BC;$$

and it is divided *externally* in extreme and mean ratio at the point D if

$$AB : AD = AD : BD.$$

Proposition XXXIX. Problem.

296. *To divide a given straight line in extreme and mean ratio* (§ 295).



Let AB be the given straight line.

To divide AB in extreme and mean ratio.

Draw BE perpendicular to AB, and equal to $\frac{1}{2}$ AB.

With E as a centre, and EB as a radius, describe the circumference BFG.

Draw the straight line AE cutting the circumference at F and G.

On AB take $AC = AF$; on BA produced take $AD = AG$.

Then AB is divided at C internally, and at D externally, in extreme and mean ratio.

For since AG is a secant, and AB a tangent,

$$AG : AB = AB : AF. \qquad (§ 283.)$$

That is, $$AG : AB = AB : AC. \qquad (1)$$

Then, $$AG - AB : AB = AB - AC : AC. \qquad (§ 237.)$$

Whence, $$AB : AG - AB = AC : BC. \qquad (§ 235.)$$

But by hypothesis,

$$AB = 2\,BE = FG. \qquad (2)$$

Whence, $$AG - AB = AG - FG$$
$$= AF = AC.$$

Substituting, we have

$$AB : AC = AC : BC. \qquad (3)$$

Therefore AB is divided at C internally in extreme and mean ratio.

Again, from (1),

$$AG + AB : AG = AB + AC : AB. \qquad (§ 236.)$$

But, $$AG + AB = AD + AB = BD.$$

And by (2), $$AB + AC = FG + AF = AG.$$

Therefore, $$BD : AG = AG : AB.$$

Whence, $$AB : AG = AG : BD. \qquad (§ 235.)$$

That is, $$AB : AD = AD : BD.$$

Therefore AB is divided at D externally in extreme and mean ratio.

297. Cor. If AB be denoted by m, and AC by x, the proportion (3) of the preceding article becomes

$$m : x = x : m - x.$$

Placing the product of the means equal to the product of the extremes, we have

$$x^2 = m^2 - mx, \qquad (§ 231.)$$

or, $$x^2 + mx = m^2.$$

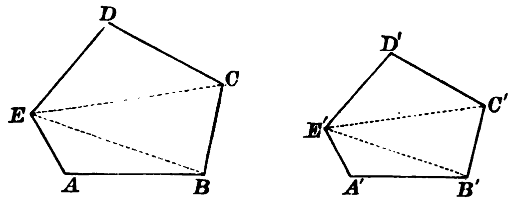
Solving this quadratic equation, we obtain

$$x = AC = \frac{m(\sqrt{5} - 1)}{2}.$$

Ex. 74. Construct a line equal to $\sqrt{2}$; to $\sqrt{5}$; to $\sqrt{6}$.

PROPOSITION XL. PROBLEM.

298. ***Upon a given side, homologous to a given side of a given polygon, to construct a polygon similar to the given polygon.***



Let $ABCDE$ be the given polygon, and $A'B'$ the given side.

To construct upon the side $A'B'$, homologous to AB, a polygon similar to $ABCDE$.

Divide the polygon $ABCDE$ into triangles by drawing the diagonals EB and EC.

At A' construct $\angle B'A'E' = \angle A$, and draw $B'E'$ making $\angle A'B'E' = \angle ABE$.

Then the triangle $A'B'E'$ will be similar to ABE. (§ 256.)

In like manner, construct the triangle $B'C'E'$ similar to BCE, and the triangle $C'D'E'$ similar to CDE.

Then $A'B'C'D'E'$ will be similar to $ABCDE$.

For two polygons are similar when they are composed of the same number of triangles, similar each to each, and similarly placed (§ 266).

EXERCISES.

75. To inscribe in a given circle a triangle similar to a given triangle.

76. To circumscribe about a given circle a triangle similar to a given triangle.

NOTE. For additional exercises on Book III., see p. 226.

BOOK IV.

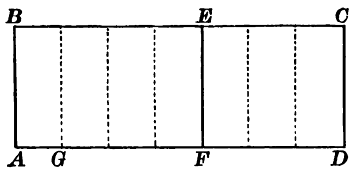
AREAS OF POLYGONS.

PROPOSITION I. THEOREM.

299. *Two rectangles having equal altitudes are to each other as their bases.*

NOTE. The word "*rectangle*," in the above statement, signifies the *amount of surface* of the rectangle.

CASE I. *When the bases are commensurable.*



Let $ABCD$ and $ABEF$ be two rectangles, having equal altitudes, and commensurable bases AD and AF.

To prove $$\frac{ABCD}{ABEF} = \frac{AD}{AF}.$$

Let AG be a common measure of AD and AF, and let it be contained 7 times in AD, and 4 times in AF.

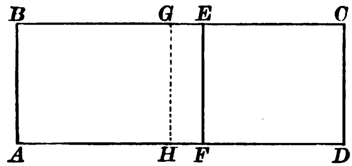
Then, $$\frac{AD}{AF} = \frac{7}{4}. \qquad (1)$$

Drawing perpendiculars to AD through the several points of division, the rectangle $ABCD$ will be divided into 7 equal parts, of which $ABEF$ will contain 4. (§ 113.)

Then, $$\frac{ABCD}{ABEF} = \frac{7}{4}. \qquad (2)$$

From (1) and (2), $$\frac{ABCD}{ABEF} = \frac{AD}{AF}.$$

CASE II. *When the bases are incommensurable.*



Let $ABCD$ and $ABEF$ be two rectangles, having equal altitudes, and incommensurable bases AD and AF.

To prove $$\frac{ABCD}{ABEF}=\frac{AD}{AF}.$$

Let AD be divided into any number of equal parts, and let one of these parts be applied to AF as a measure.

Since AD and AF are incommensurable, a certain number of the parts will extend from A to H, leaving a remainder, HF, less than one of the parts.

Draw GH perpendicular to AD.

Then, $$\frac{ABCD}{ABGH}=\frac{AD}{AH}. \qquad \text{(§ 299, Case I.)}$$

Now let the number of subdivisions of AD be indefinitely increased.

Then the length of each part will be indefinitely diminished, and the remainder HF will approach the limit 0.

Then, $\frac{ABCD}{ABGH}$ will approach the limit $\frac{ABCD}{ABEF}$,

and $\frac{AD}{AH}$ will approach the limit $\frac{AD}{AF}$.

By the Theorem of Limits, these limits are equal. (§ 188.)

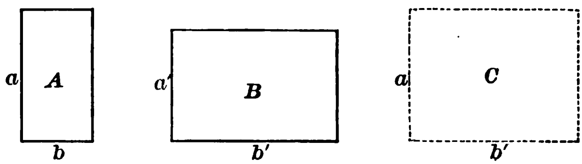
Whence, $$\frac{ABCD}{ABEF}=\frac{AD}{AF}.$$

300. COR. Since either side of a rectangle may be taken as the base, it follows that

Two rectangles having equal bases are to each other as their altitudes.

Proposition II. Theorem.

301. *Any two rectangles are to each other as the products of their bases by their altitudes.*



Let A and B be any two rectangles, having the altitudes a and a', and the bases b and b', respectively.

To prove $$\frac{A}{B} = \frac{a \times b}{a' \times b'}.$$

Let C be a rectangle having the altitude a and the base b'.

Then the rectangles A and C, having equal altitudes, are to each other as their bases. (§ 299.)

That is, $$\frac{A}{C} = \frac{b}{b'}. \qquad (1)$$

And the rectangles C and B, having equal bases, are to each other as their altitudes. (§ 300.)

That is, $$\frac{C}{B} = \frac{a}{a'}. \qquad (2)$$

Multiplying (1) and (2),

$$\frac{A}{B} = \frac{a \times b}{a' \times b'}.$$

302. Sch. It must be observed that the *product of two lines* signifies the product of their *numerical measures* when referred to a common unit.

Ex. 1. A rectangular field is 182 feet long, and 102 feet wide. Another is 119 feet long, and 117 feet wide. What is the ratio of their surfaces?

DEFINITIONS.

303. The *area* of a surface is its ratio to another surface, called the *unit of surface*, adopted arbitrarily as the unit of measure (§ 179).

Thus, if A represents a certain surface, and B the unit of surface, the area of A is $\frac{A}{B}$.

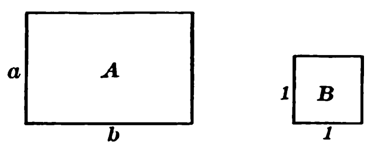
The usual unit of surface is the square whose side is some *linear unit*; for example, a *square inch*, or a *square foot*.

304. Two surfaces are said to be *equivalent* when their areas are equal.

The symbol ⇔ is used for the words "is equivalent to."

PROPOSITION III. THEOREM.

305. *The area of a rectangle is equal to the product of its base and altitude.*



Let a and b be the altitude and base of the rectangle A; and let B be the unit of surface; i.e., a square whose side is the linear unit.

To prove area of $A = a \times b$.

Any two rectangles are to each other as the products of their bases by their altitudes (§ 301).

Whence, $$\frac{A}{B} = \frac{a \times b}{1 \times 1} = a \times b.$$

But since B is the unit of surface, $\frac{A}{B}$ is the area of A. (§ 303.)

Therefore, area of $A = a \times b$.

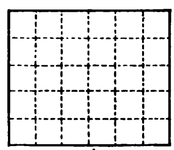
306. Cor. *The area of a square is equal to the square of its side.*

307. Sch. I. The statement of Prop. III. is an abbreviation of the following:

If the unit of surface is the square whose side is the linear unit, the *number* which expresses the area of a rectangle is equal to the product of the *numbers* which express the lengths of its sides.

An interpretation of the above form is always understood in every proposition relating to areas.

308. Sch. II. If the sides of the rectangle are *multiples* of the linear unit, the truth of Prop. III. may be seen by dividing the figure into squares, each equal to the unit of surface.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Thus, if the altitude of the rectangle A is 5 units, and its base 6 units, the figure can evidently be divided into 30 squares.

In this case, 30, the number which expresses the area of the rectangle, is the product of 6 and 5, the numbers which express the lengths of the sides.

309. Def. The *dimensions* of a rectangle are its base and altitude.

EXERCISES.

2. If the area of a rectangle is 7956 sq. in., and its base $3\frac{1}{4}$ yd., find its perimeter in feet.

3. If the base and altitude of a rectangle are 14 ft. 7 in., and 5 ft. 3 in., respectively, what is the side of an equivalent square?

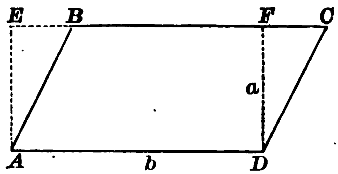
4. Find the dimensions of a rectangle whose area is 168, and perimeter 52.

5. The area of a rectangle is 143 sq. ft. 75 sq. in., and its base is 3 times its altitude. Find each of its dimensions.

6. The area of a square is 693 sq. yd. 4 sq. ft.; find its side.

Proposition IV. Theorem.

310. *The area of a parallelogram is equal to the product of its base and altitude.*



Let $ABCD$ be a parallelogram, having its altitude DF equal to a, and its base AD equal to b.

To prove area $ABCD = a \times b$.

Draw AE perpendicular to AD, meeting CB produced at E, forming the rectangle $AEFD$.

Then in the right triangles ABE and DCF,

$AB = DC$, and $AE = DF$. (§ 104.)

Whence, $\triangle ABE = \triangle DCF$. (§ 88.)

Now, if from the entire figure $AECD$ we take the triangle ABE, there remains the parallelogram $ABCD$; and if we take the triangle DCF, there remains the rectangle $AEFD$.

Therefore, area $ABCD =$ area $AEFD$.

But, area $AEFD = a \times b$. (§ 305.)

Whence, area $ABCD = a \times b$.

311. Cor. I. *Two parallelograms having equal bases and equal altitudes are equivalent* (§ 304).

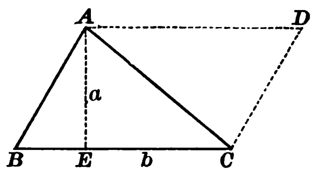
312. Cor. II. 1. *Two parallelograms having equal altitudes are to each other as their bases.*

2. *Two parallelograms having equal bases are to each other as their altitudes.*

3. *Any two parallelograms are to each other as the products of their bases by their altitudes.*

PROPOSITION V. THEOREM.

313. *The area of a triangle is equal to one-half the product of its base and altitude.*



Let ABC be a triangle, having its altitude AE equal to a, and its base BC equal to b.

To prove area $ABC = \frac{1}{2}\, a \times b$.

Draw AD and CD parallel to BC and AB, respectively, forming the parallelogram $ABCD$.

Now, $\triangle ABC = \triangle ACD$. (§ 106.)

Whence, area $ABC = \frac{1}{2}$ area $ABCD$.

But, area $ABCD = a \times b$. (§ 310.)

Whence, area $ABC = \frac{1}{2}\, a \times b$.

314. COR. I. *Two triangles having equal bases and equal altitudes are equivalent.*

315. COR. II. 1. *Two triangles having equal altitudes are to each other as their bases.*

2. *Two triangles having equal bases are to each other as their altitudes.*

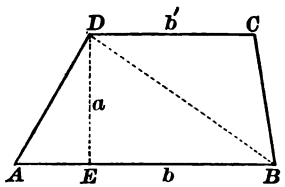
3. *Any two triangles are to each other as the products of their bases by their altitudes.*

316. COR. III. *A triangle is equivalent to one-half of a parallelogram having the same base and altitude.*

Ex. 7. The hypotenuse of a right triangle is 5 ft. 5 in., and one of its legs is 2 ft. 9 in. Find its area.

PROPOSITION VI. THEOREM.

317. *The area of a trapezoid is equal to one-half the sum of its bases multiplied by its altitude.*



Let $ABCD$ be a trapezoid, having its altitude DE equal to a, and its bases AB and DC equal to b and b', respectively.

To prove area $ABCD = a \times \frac{1}{2}(b + b')$.

Draw the diagonal BD.

Then the trapezoid is divided into two triangles, ABD and BCD, having the common altitude a.

Whence, area $ABD = \frac{1}{2} a \times b$,

and area $BCD = \frac{1}{2} a \times b'$. (§ 313.)

Adding, we have

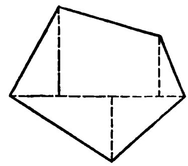
$$\text{area } ABCD = \tfrac{1}{2} a \times b + \tfrac{1}{2} a \times b'$$
$$= a \times \tfrac{1}{2}(b + b').$$

318. COR. Since the line joining the middle points of the non-parallel sides of a trapezoid is equal to one-half the sum of the bases (§ 132), it follows that

The area of a trapezoid is equal to the product of its altitude by the line joining the middle points of its non-parallel sides.

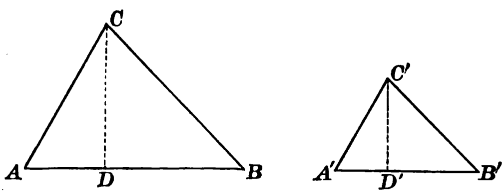
319. SCH. The area of any polygon may be obtained by finding the sum of the areas of the triangles into which the polygon may be divided by drawing diagonals from any one of its vertices.

But in practice it is better to draw the longest diagonal, and draw perpendiculars to it from the remaining vertices of the polygon. The polygon will then be divided into right triangles and trapezoids; and by measuring the lengths of the perpendiculars, and of the portions of the diagonal which they intercept, the areas of the figures may be found by §§ 313 and 317.



Proposition VII. Theorem.

320. *Two similar triangles are to each other as the squares of their homologous sides.*



Let AB and $A'B'$ be homologous sides of the similar triangles ABC and $A'B'C'$.

To prove $$\frac{ABC}{A'B'C'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

Draw the altitudes CD and $C'D'$.

Then, $$\frac{ABC}{A'B'C'} = \frac{AB \times CD}{A'B' \times C'D'} \qquad (\text{§ 315, 3.})$$

$$= \frac{AB}{A'B'} \times \frac{CD}{C'D'}. \qquad (1)$$

But, $$\frac{CD}{C'D'} = \frac{AB}{A'B'}. \qquad (\text{§ 263.})$$

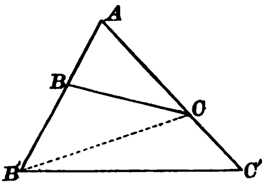
Substituting this value in (1), we have

$$\frac{ABC}{A'B'C'} = \frac{AB}{A'B'} \times \frac{AB}{A'B'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

321. SCH. Two similar triangles are to each other as the squares of any two homologous lines.

PROPOSITION VIII. THEOREM.

322. *Two triangles having an angle of one equal to an angle of the other, are to each other as the products of the sides including the equal angles.*



Let the triangles ABC and $AB'C'$ have the common angle A.

To prove $$\frac{ABC}{AB'C'} = \frac{AB \times AC}{AB' \times AC'}.$$

Draw $B'C$.

Then the triangles ABC and $AB'C$, having the common vertex C, and their bases AB and AB' in the same straight line, have the same altitude.

Whence, $$\frac{ABC}{AB'C} = \frac{AB}{AB'}. \qquad (\S\ 315,\ 1.)$$

In like manner, $$\frac{AB'C}{AB'C'} = \frac{AC}{AC'}.$$

Multiplying these equations, we have
$$\frac{ABC}{AB'C'} = \frac{AB \times AC}{AB' \times AC'}.$$

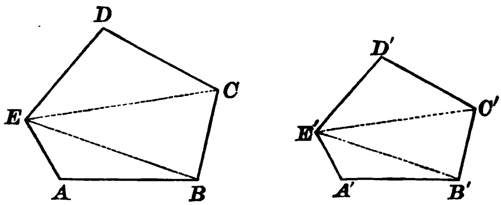
EXERCISES.

8. If the altitude of a trapezoid is 1 ft. 4 in., and its bases are 1 ft. 1 in. and 2 ft. 5 in., respectively, what is its area?

9. If, in the figure of Prop. VII., $AB = 9$, $A'B' = 7$, and the area of $A'B'C'$ is 147, what is the area of ABC?

Proposition IX. Theorem.

323. *Two similar polygons are to each other as the squares of their homologous sides.*



Let AB and $A'B'$ be homologous sides of the similar polygons AC and $A'C'$, whose areas are K and K', respectively.

To prove $$\frac{K}{K'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

Draw the diagonals EB, EC, $E'B'$, and $E'C'$.

Then the triangle ABE is similar to $A'B'E'$. (§ 267.)

Whence, $$\frac{ABE}{A'B'E'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}. \qquad (§ 320.)$$

In like manner,
$$\frac{BCE}{B'C'E'} = \frac{\overline{BC}^2}{\overline{B'C'}^2} = \frac{\overline{AB}^2}{\overline{A'B'}^2},$$

and $$\frac{CDE}{C'D'E'} = \frac{\overline{CD}^2}{\overline{C'D'}^2} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

Whence, $$\frac{ABE}{A'B'E'} = \frac{BCE}{B'C'E'} = \frac{CDE}{C'D'E'}. \qquad \text{(Ax. 1.)}$$

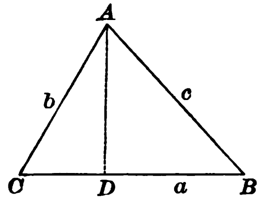
Then, $$\frac{ABE + BCE + CDE}{A'B'E' + B'C'E' + C'D'E'} = \frac{ABE}{A'B'E'}.$$
(§ 239.)

Therefore, $$\frac{K}{K'} = \frac{ABE}{A'B'E'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

324. Sch. Two similar polygons are to each other as the squares of any two homologous lines.

PROPOSITION X. PROBLEM.

325. *To find the area of a triangle when its three sides are given.*



Let a, b, and c represent the sides BC, CA, and AB, and K the area, of the triangle ABC.

To find K in terms of a, b, and c.

Let C be an acute angle, and draw AD perpendicular to BC.

Then, $$c^2 = a^2 + b^2 - 2a \times CD. \quad (\S\ 277.)$$

Transposing, $$2a \times CD = a^2 + b^2 - c^2.$$

Whence, $$CD = \frac{a^2 + b^2 - c^2}{2a}.$$

Then,

$$\overline{AD}^2 = \overline{AC}^2 - \overline{CD}^2 \quad (\S\ 274.)$$
$$= (AC + CD)(AC - CD)$$
$$= \left(b + \frac{a^2 + b^2 - c^2}{2a}\right)\left(b - \frac{a^2 + b^2 - c^2}{2a}\right)$$
$$= \frac{(2ab + a^2 + b^2 - c^2)(2ab - a^2 - b^2 + c^2)}{4a^2}$$
$$= \frac{[(a+b)^2 - c^2][c^2 - (a-b)^2]}{4a^2}$$
$$= \frac{(a+b+c)(a+b-c)(c+a-b)(c-a+b)}{4a^2}. \quad (1)$$

Now let $a + b + c = 2s$.

Then,

$$a + b - c = a + b + c - 2c = 2s - 2c = 2(s - c).$$

In like manner,

$$c + a - b = 2(s - b), \text{ and } c - a + b = 2(s - a).$$

Substituting in (1), we have

$$\overline{AD}^2 = \frac{16\, s\,(s-a)\,(s-b)\,(s-c)}{4\,a^2}.$$

Whence,
$$AD = \frac{2\sqrt{s\,(s-a)\,(s-b)\,(s-c)}}{a}.$$

Then,
$$K = \tfrac{1}{2}\, a \times AD \qquad (\S\ 313.)$$
$$= \sqrt{s\,(s-a)\,(s-b)\,(s-c)}.$$

EXERCISES.

10. The sides of a triangle are 13, 14, and 15; find its area.

SOLUTION.

Let $a = 13$, $b = 14$, and $c = 15$.
Then, $s = \frac{1}{2}(13 + 14 + 15) = 21$.
Whence, $s - a = 8$, $s - b = 7$, and $s - c = 6$.
Therefore, the area of the triangle is

$$\sqrt{21 \times 8 \times 7 \times 6} = \sqrt{3 \times 7 \times 2^3 \times 7 \times 2 \times 3}$$
$$= \sqrt{2^4 \times 3^2 \times 7^2}$$
$$= 2^2 \times 3 \times 7 = 84, \textit{ Ans.}$$

11. If the sides of a triangle ABC are $AB = 25$, $BC = 17$, and $CA = 28$, find its area, and the length of the perpendicular from each vertex to the opposite side.

12. Find the length of the diagonal of a rectangle whose area is 2640, and altitude 48.

13. Find the lower base of a trapezoid whose area is 9408, upper base 79, and altitude 96.

14. The area of a rhombus is equal to one-half the product of its diagonals.

15. The diagonals of a parallelogram divide it into four equivalent triangles.

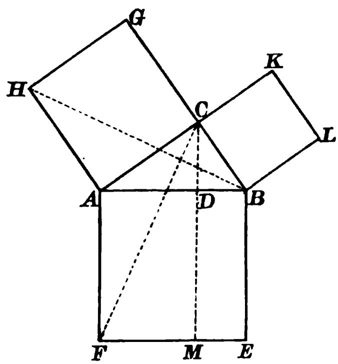
16. Lines drawn to the vertices of a parallelogram from any point in one of its diagonals divide the figure into two pairs of equivalent triangles.

17. If EF is any straight line drawn through the centre of the parallelogram $ABCD$, terminating in the sides AD and BC, prove that the triangles BEF and CED are equivalent.

326. Since the area of a square is equal to the square of its side (§ 306), we may state Prop. XXVII., Book III., as follows:

In any right triangle, the square described upon the hypotenuse is equivalent to the sum of the squares described upon the legs.

The theorem in the above form is proved as follows:



Let AB be the hypotenuse of the right triangle ABC.

To prove that the square $ABEF$, described upon AB, is equivalent to the sum of the squares $ACGH$ and $BCKL$, described upon AC and BC, respectively.

Draw CD perpendicular to AB.

Produce CD to meet EF at M, and draw BH and CF.

Then in the triangles ABH and ACF,

$$AB = AF, \text{ and } AH = AC.$$

Also, $\angle BAH = \angle CAF$,

since each is equal to a right angle $+ \angle BAC$.

Hence, $\triangle ABH = \triangle ACF$. (§ 63.)

Now the triangle ABH has the same base and altitude as the square $ACGH$.

Whence, area $ABH = \frac{1}{2}$ area $ACGH$. (§ 316.)

Again, the triangle ACF has the same base and altitude as the rectangle $ADMF$.

Whence, area $ACF = \frac{1}{2}$ area $ADMF$.

But, area $ABH =$ area ACF.

Whence, area $ACGH =$ area $ADMF$. (1)

In like manner, by drawing AL and CE, we may prove

area $BCKL =$ area $BDME$. (2)

Adding (1) and (2), we have

area $ACGH +$ area $BCKL =$ area $ABEF$.

327. Sch. The above theorem is supposed to have been first given by Pythagoras, and is called after him the *Pythagorean Theorem.*

Several other propositions of Book III. may be put in the form of statements in regard to areas; as, for example, Props. XXVIII. and XXIX.

EXERCISES.

18. The side of an equilateral triangle is 5 ; find its area.

19. The altitude of an equilateral triangle is 3; find its area.

20. The area of a certain triangle is $2\frac{1}{4}$ times the area of a similar triangle. If the altitude of the first triangle is 4 ft. 3 in., what is the homologous altitude of the second?

21. Two triangles are equivalent when they have two sides of one equal respectively to two sides of the other, and the included angles supplementary.

22. One diagonal of a rhombus is five-thirds of the other, and the difference of the diagonals is 8; find its area.

23. If D and E are the middle points of the sides BC and AC of the triangle ABC, prove that the triangles ABD and ABE are equivalent.

24. If E is the middle point of CD, one of the non-parallel sides of the trapezoid $ABCD$, and a parallel to AB drawn through E meets BC at F and AD at G, prove that the parallelogram $ABFG$ is equivalent to the trapezoid.

25. The sides AB, BC, CD, and DA of the quadrilateral $ABCD$ are 10, 17, 13, and 20, respectively, and the diagonal AC is 21. Find the area of the quadrilateral.

26. Find the area of the square inscribed in a circle whose radius is 3.

27. The area of an isosceles right triangle is 81 sq. in.; find its hypotenuse in feet.

28. The area of an equilateral triangle is $9\sqrt{3}$; find its side.

29. The area of an equilateral triangle is $16\sqrt{3}$; find its altitude.

30. The base of an isosceles triangle is 56, and each of the equal sides is 53; find its area.

31. The area of a triangle is equal to one-half the product of its perimeter by the radius of the inscribed circle.

32. The area of an isosceles right triangle is equal to one-fourth the area of the square described upon the base.

33. If the angle A of the triangle ABC is $30°$, prove that

$$\text{area } ABC = \tfrac{1}{4} AB \times AC.$$

34. A circle whose diameter is 12 is inscribed in a quadrilateral whose perimeter is 40. Find the area of the quadrilateral.

35. Two similar triangles have homologous sides equal to 8 and 15, respectively. Find the homologous side of a similar triangle equivalent to their sum.

36. The non-parallel sides of a trapezoid are each 25 units in length, and the parallel sides are 19 and 33 units, respectively. Find the area of the trapezoid.

37. If E is any point within the parallelogram $ABCD$, the triangles ABE and CDE are together equivalent to one-half the parallelogram.

38. If the area of a polygon, one of whose sides is 15 in., is 375 sq. in., what is the area of a similar polygon whose homologous side is 18 in.?

39. If the area of a polygon, one of whose sides is 36 ft., is 648 sq. ft., what is the homologous side of a similar polygon whose area is 392 sq. ft.?

40. If one diagonal of a quadrilateral bisects the other, it divides the quadrilateral into two equivalent triangles.

41. Two equivalent triangles have a common base, and lie on opposite sides of it. Prove that the base, produced if necessary, bisects the line joining their vertices.

42. If the sides of a triangle are 15, 41, and 52, find the radius of the inscribed circle. (Ex. 31.)

43. The area of a rhombus is 240, and its side is 17; find its diagonals.

44. If the sides of a triangle are 25, 29, and 36, find the diameter of the circumscribed circle. (§ 287.)

45. The sum of the perpendiculars from any point within an equilateral triangle to the three sides is equal to the altitude of the triangle.

46. The longest sides of two similar polygons are 18 and 3, respectively. How many polygons, each equal to the second, will form a polygon equivalent to the first?

47. The sides AB and AC of a triangle ABC are 15 and 22, respectively. From a point D in AB, a parallel to BC is drawn meeting AC at E, and dividing the triangle into two equivalent parts. Find AD and AE.

48. The segments of the hypotenuse of a right triangle made by a perpendicular drawn from the vertex of the right angle, are $5\frac{2}{5}$ and $9\frac{3}{5}$; find the area of the triangle.

49. Any straight line drawn through the centre of a parallelogram, terminating in a pair of opposite sides, divides the parallelogram into two equivalent quadrilaterals.

50. If E is the middle point of CD, one of the non-parallel sides of the trapezoid $ABCD$, prove that the triangle ABE is equivalent to one-half the trapezoid.

51. The sides of a triangle ABC are $AB = 13$, $BC = 14$, and $CA = 15$. If AD is the bisector of the angle A, find the areas of the triangles ABD and ACD.

52. The longest diagonal AD of a pentagon $ABCDE$ is 44, and the perpendiculars to it from B, C, and E are 24, 16, and 15, respectively. If $AB = 25$, $CD = 20$, and $AE = 17$, what is the area of the pentagon?

53. The sides of a triangle are proportional to the numbers 7, 24, and 25. The perpendicular to the third side from the vertex of the opposite angle is $13\frac{11}{25}$. Find the area of the triangle.

54. If E and F are the middle points of the sides AB and AC of a triangle, and D is any point in BC, prove that the quadrilateral $AEDF$ is equivalent to one-half the triangle ABC.

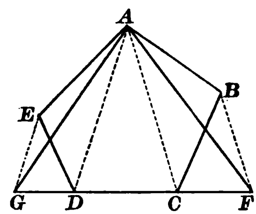
55. The parallelogram formed by joining the middle points of the adjacent sides of a quadrilateral is equivalent to one-half the quadrilateral.

NOTE. For additional exercises on Book IV., see p. 226.

PROBLEMS IN CONSTRUCTION.

PROPOSITION XI. PROBLEM.

328. *To construct a triangle equivalent to a given polygon.*



Let $ABCDE$ be the given polygon.

To construct a triangle equivalent to $ABCDE$.

Let A, B, and C be any three consecutive vertices, and draw the diagonal AC.

Draw BF parallel to AC, meeting DC produced at F, and draw AF.

Then $AFDE$ is a polygon equivalent to $ABCDE$, having a number of sides less by one.

For the triangles ABC and AFC have the same base AC.

And since their vertices B and F lie in the same straight line parallel to AC, they have the same altitude. (§ 78.)

Therefore, area ABC = area AFC. (§ 314.)

Adding area $ACDE$ to both members, we have

area $ABCDE$ = area $AFDE$.

Again, draw the diagonal AD.

Draw EG parallel to AD, meeting CD produced at G, and draw AG.

Then, area AED = area AGD. (§ 314.)

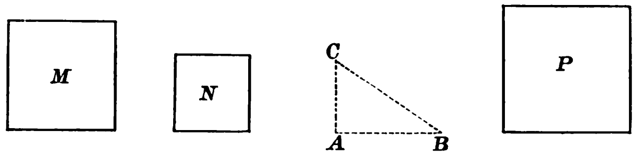
Adding area AFD to both members, we have

area $AFDE$ = area AFG.

Whence, area $ABCDE$ = area AFG. (Ax. 1.)

Proposition XII. Problem.

329. *To construct a square equivalent to the sum of two given squares.*



Let M and N be the given squares.

To construct a square equivalent to the sum of M and N.

Draw AB equal to a side of M; at A draw AC perpendicular to AB, and equal to a side of N, and draw BC.

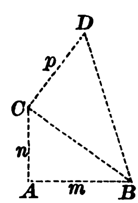
Then the square P, described with its side equal to BC, will be equivalent to the sum of M and N.

For in the right triangle ABC,

$$\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2. \qquad (\S\ 273.)$$

That is, area $P =$ area $M +$ area N. (§ 306.)

330. Cor. By an extension of the above method, a square may be constructed equivalent to the sum of any number of given squares.



Let it be required, for example, to construct a square equivalent to the sum of three squares, whose sides are m, n, and p, respectively.

Draw AB equal to m.

At A draw AC perpendicular to AB, and equal to n, and draw BC.

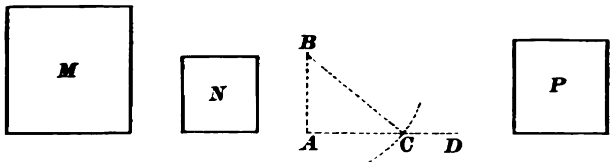
At C draw CD perpendicular to BC, and equal to p, and draw BD.

Then the square described with its side equal to BD will be equivalent to the sum of the given squares.

For, $\overline{BD}^2 = \overline{BC}^2 + p^2 = m^2 + n^2 + p^2$. (§ 273.)

PROPOSITION XIII. PROBLEM.

331. *To construct a square equivalent to the difference of two given squares.*



Let M and N be the given squares.

To construct a square equivalent to the difference of M and N.

Draw the indefinite straight line AD.

At A draw AB perpendicular to AD, and equal to a side of N, the smaller of the given squares.

With B as a centre, and with a radius equal to a side of M, describe an arc cutting AD at C.

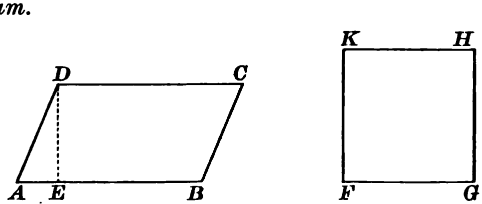
Then the square P, described with its side equal to AC, will be equivalent to the difference of M and N.

For, $\overline{AC}^2 = \overline{BC}^2 - \overline{AB}^2$. (§ 274.)

That is, area P = area M — area N. (§ 306.)

PROPOSITION XIV. PROBLEM.

332. *To construct a square equivalent to a given parallelogram.*



Let $ABCD$ be the given parallelogram.

To construct a square equivalent to $ABCD$.

Draw DE perpendicular to AB, and construct FG a mean proportional between AB and DE (§ 293).

Then the square $FGHK$, described with its side equal to FG, will be equivalent to $ABCD$.

For by construction,

$$AB : FG = FG : DE.$$

Whence, $\overline{FG}^2 = AB \times DE.$ (§ 231.)

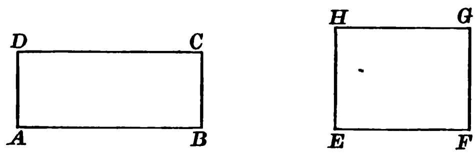
That is, area $FGHK$ = area $ABCD$. (§§ 306, 310.)

333. Cor. *A square may be constructed equivalent to a given triangle by taking for its side a mean proportional between the base and one-half the altitude of the triangle.*

334. Sch. By aid of §§ 328 and 333, a square may be constructed equivalent to a given polygon.

PROPOSITION XV. PROBLEM.

335. *With a given straight line as a base, to construct a rectangle equivalent to a given rectangle.*



Let $ABCD$ be the given rectangle, and EF the given line.

To construct, with EF as a base, a rectangle equivalent to $ABCD$.

The rectangle $EFGH$, constructed with EF as a base, and with its side EH equal to a fourth proportional to EF, AB, and AD (§ 291), will be equivalent to $ABCD$.

For by construction,

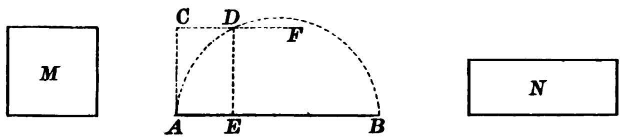
$$EF : AB = AD : EH.$$

Whence, $EF \times EH = AB \times AD.$ (§ 231.)

That is, area $EFGH$ = area $ABCD$. (§ 305.)

PROPOSITION XVI. PROBLEM.

336. *To construct a rectangle equivalent to a given square, having the sum of its base and altitude equal to a given line.*



Let M be the given square, and AB the given line.

To construct a rectangle equivalent to M, having the sum of its base and altitude equal to AB.

With AB as a diameter, describe the semi-circumference ADB.

Draw AC perpendicular to AB, and equal to a side of M.

Draw CF parallel to AB, intersecting the arc ADB at D, and draw DE perpendicular to AB.

Then the rectangle N, constructed with its base and altitude equal to BE and AE, respectively, will be equivalent to M.

For, $AE : DE = DE : BE$. (§ 272, I.)

Whence, $\overline{DE}^2 = AE \times BE$. (§ 231.)

That is, area M = area N. (§§ 306, 305.)

NOTE. The above construction is also the solution of the following problem:

Given the sum and the product of two straight lines, to construct the lines.

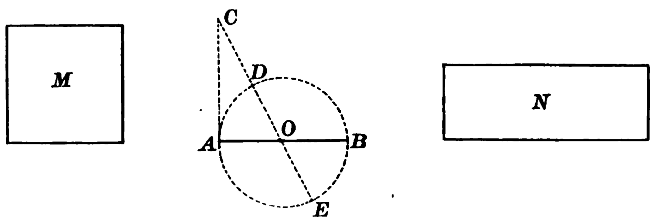
EXERCISES.

56. To construct a triangle equivalent to a given triangle, having given its base.

57. To construct a triangle equivalent to a given square, having given its base and an angle adjacent to the base.

PROPOSITION XVII. PROBLEM.

337. *To construct a rectangle equivalent to a given square, having the difference of its base and altitude equal to a given line.*



Let M be the given square, and AB the given line.

To construct a rectangle equivalent to M, having the difference of its base and altitude equal to AB.

With the line AB as a diameter, describe the circumference ADB.

Draw AC perpendicular to AB, and equal to a side of M.

Through the centre O draw CO, intersecting the circumference at D and E.

Then the rectangle N, constructed with its base and altitude equal to CE and CD, respectively, will be equivalent to M.

For, $$CE - CD = DE = AB.$$

That is, the difference of the base and altitude of N is equal to AB.

Again, AC is tangent to the circle at A. (§ 169.)

Whence, $\overline{CA}^2 = CD \times CE.$ (§ 282.)

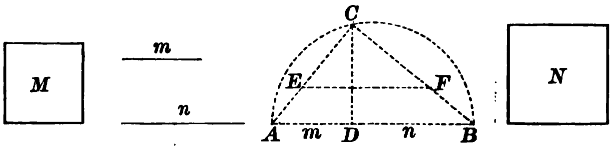
That is, area M = area N. (§§ 306, 305.)

NOTE. The above construction is also the solution of the following problem:

Given the difference and the product of two straight lines, to construct the lines.

Proposition XVIII. Problem.

338. *To construct a square having a given ratio to a given square.*



Let M be the given square, and let the given ratio be that of the lines m and n.

To construct a square which shall have to M the ratio $n : m$.

On the straight line AB, take $AD = m$ and $DB = n$.

With AB as a diameter, describe the semi-circumference ACB.

Draw DC perpendicular to AB, meeting the arc ACB at C, and draw AC and BC.

On AC take CE equal to a side of M; and draw EF parallel to AB, meeting BC at F.

Then the square N, described with its side equal to CF, will have to M the ratio $n : m$.

For, $\angle ACB$ is a right angle. (§ 196.)

Then since CD is perpendicular to AB,

$$\frac{\overline{AC}^2}{\overline{BC}^2} = \frac{AD}{BD} = \frac{m}{n}. \qquad (§ 271.)$$

But since EF is parallel to AB,

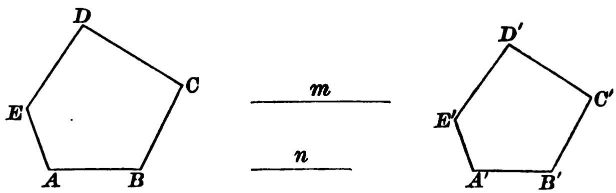
$$\frac{CE}{CF} = \frac{AC}{BC}. \qquad (§ 245.)$$

Therefore, $$\frac{\overline{CE}^2}{\overline{CF}^2} = \frac{\overline{AC}^2}{\overline{BC}^2} = \frac{m}{n}.$$

That is, $$\frac{\text{area } M}{\text{area } N} = \frac{m}{n}. \qquad (§ 306.)$$

Proposition XIX. Problem.

339. *To construct a polygon similar to a given polygon, and having a given ratio to it.*



Let A–E be the given polygon, and let the given ratio be that of the lines m and n.

To construct a polygon similar to A–E, and having to it the ratio $n : m$.

Construct $A'B'$, the side of a square which shall have to the square described upon AB the ratio $n : m$ (§ 338).

Upon the side $A'B'$, homologous to AB, construct the polygon A'–E' similar to A–E (§ 298).

Then A'–E' will have to A–E the ratio $n : m$.

For since A–E is similar to A'–E',

$$\frac{A\text{–}E}{A'\text{–}E'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}. \qquad (§ 323.)$$

But by construction,

$$\frac{\overline{AB}^2}{\overline{A'B'}^2} = \frac{m}{n}.$$

Whence,

$$\frac{A\text{–}E}{A'\text{–}E'} = \frac{m}{n}.$$

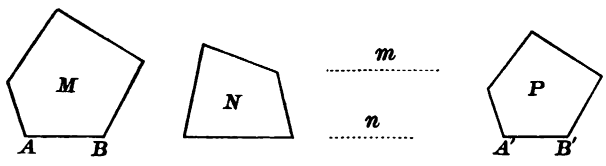
EXERCISES.

58. To construct a triangle equivalent to a given square, having given its base and the median drawn from the vertex to the base.

59. To construct a square equivalent to twice a given square.

PROPOSITION XX. PROBLEM.

340. *To construct a polygon similar to one of two given polygons, and equivalent to the other.*



Let M and N be the given polygons.

To construct a polygon similar to M, and equivalent to N.

Let AB be any side of M.

Construct m, the side of a square equivalent to M, and n, the side of a square equivalent to N (§ 334).

Construct $A'B'$ a fourth proportional to m, n, and AB (§ 291).

Upon the side $A'B'$, homologous to AB, construct the polygon P similar to M (§ 298).

Then P will be equivalent to N.

For since M is similar to P,

$$\frac{\text{area } M}{\text{area } P} = \frac{\overline{AB}^2}{\overline{A'B'}^2}. \qquad (\S\ 323.)$$

But by construction, we have

$$\frac{AB}{A'B'} = \frac{m}{n}.$$

Whence, $$\frac{\text{area } M}{\text{area } P} = \frac{m^2}{n^2}.$$

But, $$m^2 = \text{area } M,$$

and $$n^2 = \text{area } N. \qquad (\S\ 306.)$$

Whence, $$\frac{\text{area } M}{\text{area } P} = \frac{\text{area } M}{\text{area } N}.$$

Therefore, $$\text{area } P = \text{area } N.$$

EXERCISES.

60. To construct an isosceles triangle equivalent to a given triangle, having its base coincident with a side of the given triangle.

61. To construct a rhombus equivalent to a given parallelogram, having one of its diagonals coincident with a diagonal of the parallelogram.

62. To construct a triangle equivalent to a given triangle, having given two of its sides.

63. To construct a right triangle equivalent to a given square, having given its hypotenuse.

64. To construct a right triangle equivalent to a given triangle, having given its hypotenuse.

65. To construct an isosceles triangle equivalent to a given triangle, having given one of its equal sides. How many different triangles can be constructed?

66. To draw a line parallel to the base of a triangle dividing it into two equivalent parts. (§ 320.)

67. To draw through a given point within a parallelogram a straight line dividing it into two equivalent parts.

68. To construct a parallelogram equivalent to a given trapezoid, having a side and two adjacent angles equal to one of the non-parallel sides and the adjacent angles of the trapezoid.

69. To draw through a given point in a side of a parallelogram a straight line dividing it into two equivalent parts.

70. To draw a straight line perpendicular to the bases of a trapezoid, dividing the trapezoid into two equivalent parts.

(Draw a line connecting the middle points of the bases.)

71. To draw through a given point in one of the bases of a trapezoid a straight line dividing the trapezoid into two equivalent parts.

72. To construct a triangle similar to two given similar triangles, and equivalent to their sum.

73. To construct a triangle similar to two given similar triangles, and equivalent to their difference.

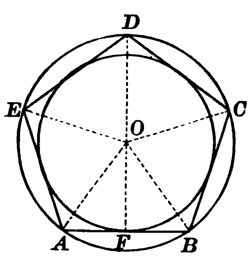
BOOK V.

REGULAR POLYGONS. — MEASUREMENT OF THE CIRCLE.

341. DEF. A *regular polygon* is a polygon which is both equilateral and equiangular.

PROPOSITION I. THEOREM.

342. *A circle can be circumscribed about, or inscribed in, any regular polygon.*



Let $ABCDE$ be a regular polygon.

I. To prove that a circle can be circumscribed about $ABCDE$.

Let a circumference be described through the vertices A, B, and C (§ 223).

Let O be the centre of the circumference, and draw OA, OB, OC, and OD.

Then since $ABCDE$ is equiangular,

$$\angle ABC = \angle BCD.$$

And since the triangle OBC is isosceles,

$$\angle OBC = \angle OCB. \qquad (\S\ 91.)$$

Then, $\angle ABC - \angle OBC = \angle BCD - \angle OCB.$

That is, $\angle OBA = \angle OCD.$

Also, $OB = OC.$ (§ 143.)

And since $ABCDE$ is equilateral,

$$AB = CD.$$

Therefore, $\triangle OAB = \triangle OCD.$ (§ 63.)

Whence, $OA = OD.$ (§ 66.)

Then the circumference passing through A, B, and C also passes through D.

In like manner, it may be proved that the circumference passing through B, C, and D also passes through E.

Hence, a circle can be circumscribed about $ABCDE$.

II. To prove that a circle can be inscribed in $ABCDE$.

Since AB, BC, CD, etc., are equal chords of the circumscribed circle, they are equally distant from O. (§ 164.)

Hence, a circle described with O as a centre, and with the perpendicular OF from O to any side AB as a radius, will be inscribed in $ABCDE$.

343. Def. The *centre* of a regular polygon is the common centre of the circumscribed and inscribed circles.

The *angle at the centre* is the angle between the radii drawn to the extremities of any side; as AOB.

The *radius* is the radius of the circumscribed circle; as OA.

The *apothem* is the radius of the inscribed circle; as OF.

344. Cor. From the equal triangles OAB, OBC, etc., we have

$$\angle AOB = \angle BOC = \angle COD, \text{ etc.} \qquad (\S\ 66.)$$

Then each of these angles is equal to four right angles divided by the number of sides of the polygon. (§ 37.)

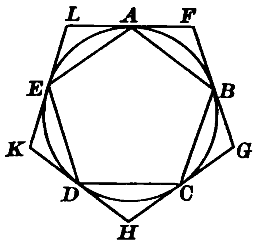
That is, *the angle at the centre of a regular polygon is equal to four right angles, divided by the number of sides.*

PROPOSITION II. THEOREM.

345. *If the circumference of a circle be divided into any number of equal arcs,*

I. *Their chords form a regular inscribed polygon.*

II. *Tangents at the points of division form a regular circumscribed polygon.*



Let the circumference ACD be divided into any number of equal arcs, AB, BC, CD, etc.

I. To prove $ABCDE$ a regular polygon.

Now, chord $AB =$ chord $BC =$ chord CD, etc. (§ 158.)

Again, since arc $AB =$ arc $BC =$ arc CD, etc., we have

$$\text{arc } BCDE = \text{arc } CDEA = \text{arc } DEAB, \text{ etc.}$$

Whence, $\angle EAB = \angle ABC = \angle BCD$, etc. (§ 193.)

Therefore, the polygon $ABCDE$ is regular. (§ 341.)

II. Let $FGHKL$ be a polygon whose sides LF, FG, etc., are tangent to the circle at the points A, B, etc., respectively.

To prove $FGHKL$ a regular polygon.

In the triangles ABF, BCG, CDH, etc., we have

$$AB = BC = CD, \text{ etc.}$$

Also, since arc $AB =$ arc $BC =$ arc CD, etc., we have

$$\angle BAF = \angle ABF = \angle CBG = \angle BCG, \text{ etc.}$$

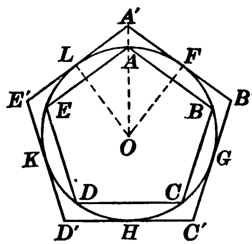
(§ 197.)

Hence, the triangles ABF, BCG, etc., are all equal and isosceles. (§§ 68, 94.)

Therefore, $\angle F = \angle G = \angle H$, etc.,
and $BF = BG = CG = CH$, etc. (§ 66.)
Whence, $FG = GH = HK$, etc.
Therefore, the polygon $FGHKL$ is regular. (§ 341.)

PROPOSITION III. THEOREM.

346. *Tangents to a circle at the middle points of the arcs subtended by the sides of a regular inscribed polygon, form a regular circumscribed polygon.*



Let $ABCDE$ be a regular polygon inscribed in the circle AC.

Let $A'B'C'D'E'$ be a polygon whose sides $A'B'$, $B'C'$, etc., are tangent to the circle at the middle points F, G, etc., of the arcs AB, BC, etc.

To prove $A'B'C'D'E'$ a regular polygon.

We have, arc AB = arc BC = arc CD, etc. (§ 157.)
Whence, arc AF = arc BF = arc BG = arc CG, etc.
Therefore, arc FG = arc GH = arc HK, etc.
Whence, the polygon $A'B'C'D'E'$ is regular. (§ 345, II.)

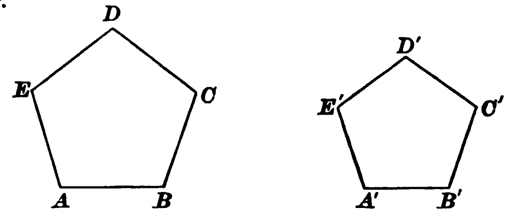
347. COR. Let O be the centre of the circle, and draw OF, OL, and OA'.

Then, OA' bisects $\angle FOL$. (§ 175.)
Whence, OA' passes through A. (§ 154.)

That is, *the radii of a regular circumscribed polygon intersect the circumference in points which are the vertices of a regular inscribed polygon having the same number of sides.*

Proposition IV. Theorem.

348. *Regular polygons of the same number of sides are similar.*



Let A–E and A'–E' be two regular polygons of the same number of sides.

To prove A–E and A'–E' similar.

The sum of all the angles of A–E is equal to the sum of all the angles of A'–E'. (§ 126.)

Whence, each angle of A–E equals each angle of A'–E'.

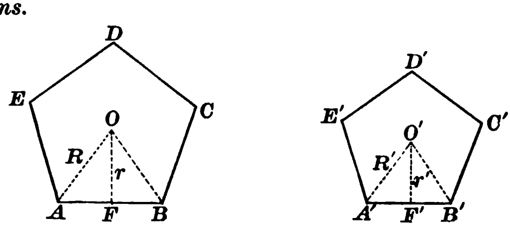
Again, since $AB = BC$, etc., and $A'B' = B'C'$, etc.,

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}, \text{ etc.}$$

Therefore, A–E and A'–E' are similar. (§ 252.)

Proposition V. Theorem.

349. *The perimeters of two regular polygons of the same number of sides are to each other as their radii, or as their apothems.*



Let A–E and A'–E' be two regular polygons of the same number of sides,

Let P and P' denote their perimeters, R and R' their radii, and r and r' their apothems.

To prove $$\frac{P}{P'} = \frac{R}{R'} = \frac{r}{r'}.$$

Let O and O' be the centres of the polygons A–E and A'–E'.

Draw OA, OB, $O'A'$, and $O'B'$.

Also, draw OF and $O'F'$ perpendicular to AB and $A'B'$, respectively.

Then, $OA = R$, $O'A' = R'$, $OF = r$, and $O'F' = r'$.

Now in the isosceles triangles OAB and $O'A'B'$,

$$\angle AOB = \angle A'O'B'. \qquad (\S\ 344.)$$

And since $OA = OB$, and $O'A' = O'B'$, we have

$$\frac{OA}{O'A'} = \frac{OB}{O'B'}.$$

Therefore, the triangles OAB and $O'A'B'$ are similar. (§ 260.)

Whence, $$\frac{AB}{A'B'} = \frac{R}{R'} = \frac{r}{r'}. \qquad (\S\S\ 253,\ \text{II.},\ 263.)$$

But the polygons A–E and A'–E' are similar. (§ 348.)

Whence, $$\frac{P}{P'} = \frac{AB}{A'B'}. \qquad (\S\ 268.)$$

Therefore, $$\frac{P}{P'} = \frac{R}{R'} = \frac{r}{r'}.$$

350. Cor. Let K and K' denote the areas of the polygons A–E and A'–E'.

Then, $$\frac{K}{K'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}. \qquad (\S\ 320.)$$

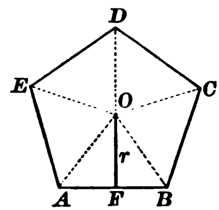
But, $$\frac{AB}{A'B'} = \frac{R}{R'} = \frac{r}{r'}.$$

Whence, $$\frac{K}{K'} = \frac{R^2}{R'^2} = \frac{r^2}{r'^2}.$$

That is, *the areas of two regular polygons of the same number of sides are to each other as the squares of their radii, or as the squares of their apothems.*

Proposition VI. Theorem.

351. *The area of a regular polygon is equal to one-half the product of its perimeter and apothem.*



Let r denote the apothem OF, and P the perimeter, of the regular polygon A–E.

To prove area $A\text{–}E = \frac{1}{2} P \times r$.

Draw the radii OA, OB, OC, etc., forming a series of triangles, OAB, OBC, etc., whose common altitude is r.

Now, area $OAB = \frac{1}{2} AB \times r$,

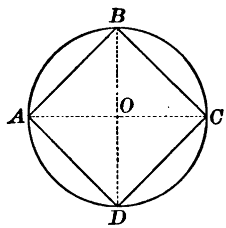
area $OBC = \frac{1}{2} BC \times r$, etc. (§ 313.)

Adding, we have

$$\text{area } A\text{–}E = \tfrac{1}{2}(AB + BC + \text{etc.}) \times r$$
$$= \tfrac{1}{2} P \times r.$$

Proposition VII. Problem.

352. *To inscribe a square in a given circle.*



Let AC be the given circle.

To inscribe a square in AC.

Draw the diameters AC and BD perpendicular to each other, and draw AB, BC, CD, and DA.

Then $ABCD$ is an inscribed square.

For, arc $AB =$ arc $BC =$ arc $CD =$ arc DA. (§ 191.)
Whence, $ABCD$ is an inscribed square. (§ 345, I.)

353. Cor. Denoting the radius OA by R, we have

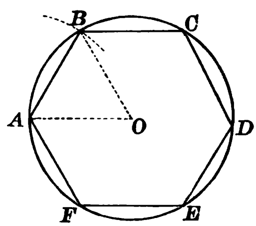
$$\overline{AB}^2 = \overline{OA}^2 + \overline{OB}^2 = 2R^2. \qquad (§ 273.)$$

Whence, $$AB = R\sqrt{2}.$$

That is, *the side of an inscribed square is equal to the radius of the circle multiplied by* $\sqrt{2}$.

Proposition VIII. Problem.

354. *To inscribe a regular hexagon in a given circle.*



Let AC be the given circle.

To inscribe a regular hexagon in AC.

Draw any radius OA.

With A as a centre, and OA as a radius, describe an arc cutting the given circumference at B, and draw AB.

Then AB is a side of a regular inscribed hexagon.

For drawing OB, the triangle OAB is equilateral.

Whence, $\angle AOB$ is one-third of two right angles. (§ 82.)

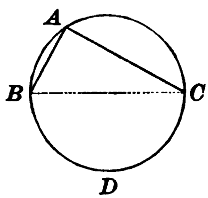
Then, $\angle AOB$ is one-sixth of four right angles, and AB is a side of a regular inscribed hexagon. (§ 345, I.)

Hence, to inscribe a regular hexagon in a circle, apply the radius six times as a chord.

355. Cor. I. *The side of a regular inscribed hexagon is equal to the radius of the circle.*

356. Cor. II. *By joining the alternate vertices of the hexagon, there is formed an inscribed equilateral triangle.*

357. Cor. III. Let AB be a side of a regular hexagon, inscribed in the circle AD whose radius is R.



Draw the diameter BC, and join AC.

Then AC is a side of an inscribed equilateral triangle.

Now ABC is a right triangle. (§ 196.)

Then, $\overline{AC}^2 = \overline{BC}^2 - \overline{AB}^2$ (§ 274.)

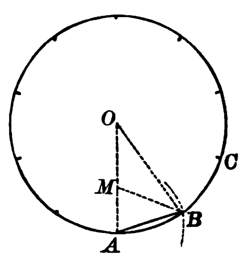
$$= (2\,R)^2 - R^2 = 4\,R^2 - R^2 = 3\,R^2.$$

Whence, $AC = R\sqrt{3}$.

That is, *the side of an inscribed equilateral triangle is equal to the radius of the circle multiplied by* $\sqrt{3}$.

Proposition IX. Problem.

358. *To inscribe a regular decagon in a given circle.*



Let AC be the given circle.

To inscribe a regular decagon in AC.

Draw any radius OA.

Divide OA in extreme and mean ratio (§ 296), so that

$$OA : OM = OM : AM. \qquad (1)$$

With A as a centre, and OM as a radius, describe an arc cutting the given circumference at B, and draw AB.

Then AB is a side of a regular inscribed decagon.

For draw OB and BM.

Then in the triangles OAB and ABM, $\angle A$ is common.

And since $OM = AB$, the proportion (1) gives

$$OA : AB = AB : AM.$$

Therefore, OAB and ABM are similar. (§ 260.)

Whence, $\angle ABM = \angle AOB.$ (2)

Now the triangle OAB is isosceles.

Hence, the similar triangle ABM is isosceles, and

$$AB = BM = OM.$$

Therefore, $\angle OBM = \angle AOB.$ (3)

Adding (2) and (3), we have

$$\angle OBA = 2 \angle AOB. \quad (4)$$

But since the triangle OAB is isosceles,

$$2 \angle OBA + \angle AOB = 180°. \quad (\S\ 82.)$$

Then by (4),

$$5 \angle AOB = 180°, \text{ or } \angle AOB = 36°.$$

Therefore, $\angle AOB$ is one-tenth of four right angles, and AB is a side of a regular inscribed decagon. (§ 345, I.)

Hence, to inscribe a regular decagon in a circle, divide the radius in extreme and mean ratio, and apply the greater segment ten times as a chord.

359. Cor. I. *By joining the alternate vertices of the decagon, there is formed a regular inscribed pentagon.*

360. Cor. II. Denoting the radius of the circle by R, we have

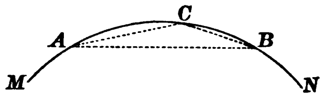
$$AB = OM = \frac{R(\sqrt{5} - 1)}{2}. \quad (\S\ 297.)$$

This is an expression for the side of a regular inscribed decagon in terms of the radius of the circle.

Ex. 1. The angle at the centre of a regular polygon is the supplement of the angle of the polygon.

Proposition X. Problem.

361. *To construct the side of a regular pentedecagon inscribed in a given circle.*



Let MN be an arc of the given circumference.

To construct the side of a regular inscribed polygon of fifteen sides.

Construct AB equal to a side of a regular inscribed hexagon (§ 354), and AC equal to a side of a regular inscribed decagon (§ 358), and draw BC.

Then BC is a side of a regular inscribed pentedecagon.

For the arc BC is $\frac{1}{6} - \frac{1}{10}$, or $\frac{1}{15}$, of the circumference.

Hence, the chord BC is a side of a regular inscribed pentedecagon. (§ 345, I.)

362. Sch. By bisecting the arcs AB, BC, etc., in the figure of Prop. VII., we may construct a regular inscribed octagon (§ 345, I.); and by continuing the bisection, we may construct regular inscribed polygons of 16, 32, 64, etc., sides.

In like manner, by aid of Props. VIII., IX., and X., we may construct regular inscribed polygons of 12, 24, 48, etc., or of 20, 40, 80, etc., or of 30, 60, 120, etc., sides.

By drawing tangents to the circumference at the vertices of any one of the above inscribed polygons, we may construct a regular circumscribed polygon of the same number of sides. (§ 345, II.)

EXERCISES.

2. An equilateral polygon inscribed in a circle is regular.

3. An equiangular polygon circumscribed about a circle is regular.

Find the angle, and the angle at the centre,

4. Of a regular pentagon.

5. Of a regular dodecagon.

6. Of a regular polygon of 32 sides.

7. Of a regular polygon of 25 sides.

If r represents the radius, a the apothem, s the side, and k the area, prove that:

8. In an equilateral triangle, $a = \frac{1}{2} r$, and $k = \frac{3}{4} r^2 \sqrt{3}$.

9. In a square, $a = \frac{1}{2} r \sqrt{2}$, and $k = 2 r^2$.

10. In a regular hexagon, $a = \frac{1}{2} r \sqrt{3}$, and $k = \frac{3}{2} r^2 \sqrt{3}$.

11. In an equilateral triangle, $r = 2a$, $s = 2a\sqrt{3}$, and $k = 3a^2\sqrt{3}$.

12. In a square, $r = a\sqrt{2}$, $s = 2a$, and $k = 4a^2$.

13. In a regular hexagon, $r = \frac{2}{3} a \sqrt{3}$, and $k = 2a^2\sqrt{3}$.

14. In an equilateral triangle, express r, a, and k in terms of s.

15. In a square, express r, a, and k in terms of s.

16. In a regular hexagon, express a and k in terms of s.

17. In an equilateral triangle, express r, a, and s in terms of k.

18. In a square, express r, a, and s in terms of k.

19. In a regular hexagon, express r and a in terms of k.

20. The apothem of an equilateral triangle is one-third the altitude of the triangle.

21. The sides of a regular polygon circumscribed about a circle are bisected at the points of contact.

22. The radius drawn from the centre of a regular polygon to any vertex bisects the angle at that vertex.

23. The diagonals of a regular pentagon are equal.

24. The figure bounded by the five diagonals of a regular pentagon is a regular pentagon.

25. The area of a regular inscribed hexagon is a mean proportional between the areas of an inscribed, and of a circumscribed equilateral triangle.

26. If the diagonals AC and BE of the regular pentagon $ABCDE$ intersect at F, prove $AC = AE + BF$. (Ex. 23.)

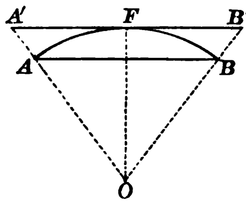
27. In the figure of Prop. VIII., prove that OM is the side of a regular pentagon inscribed in a circle which is circumscribed about the triangle OBM.

Proposition XI. Theorem.

363. ***If a regular polygon be inscribed in, or circumscribed about, a circle, and the number of its sides be indefinitely increased,***

I. ***Its perimeter approaches the circumference as a limit.***

II. ***Its area approaches the area of the circle as a limit.***



Let p and P denote the perimeters, and k and K the areas, of two regular polygons of the same number of sides, respectively inscribed in, and circumscribed about, a circle.

Let C denote the circumference of the circle, and S its area.

I. To prove that if the number of sides of the polygons be indefinitely increased, P and p approach the limit C.

Let $A'B'$ be a side of the polygon whose perimeter is P.

Draw the radius OF to its point of contact; also, draw OA' and OB' cutting the circumference at A and B, and draw AB.

Then, AB is a side of the polygon whose perimeter is p. (§ 347.)

Now the two polygons are similar. (§ 348.)

Whence, $$\frac{P}{p} = \frac{OA'}{OF}. \qquad (§ 349.)$$

Then, $$\frac{P-p}{p} = \frac{OA'-OF}{OF}. \qquad (§ 237.)$$

Or, $$P - p = \frac{p}{OF} \times (OA' - OF). \qquad (1)$$

Now let the number of sides of each polygon be indefinitely increased, the two polygons continuing to have the same number of sides.

Then the length of each side will be indefinitely diminished, and $A'F$ will approach the limit 0.

Therefore, $OA' - OF$ will approach the limit 0.

Whence, by (1), $P - p$ will approach the limit 0; that is, the difference between the perimeters of the polygons will approach the limit 0.

But the circumference of the circle is evidently less than the perimeter of the circumscribed polygon; and it is greater than the perimeter of the inscribed polygon. (Ax. 6.)

Hence, the difference between the perimeter of either polygon and the circumference of the circle will approach the limit 0.

That is, $P - C$ and $C - p$ will each approach the limit 0.

Therefore, P and p will each approach the limit C.

II. To prove that K and k approach the limit S.

Since the polygons whose sides are AB and $A'B'$ are similar,

$$\frac{K}{k} = \frac{\overline{OA'}^2}{\overline{OF}^2}. \qquad (§ 350.)$$

Whence, $$\frac{K-k}{k} = \frac{\overline{OA'}^2 - \overline{OF}^2}{\overline{OF}^2} = \frac{\overline{A'F}^2}{\overline{OF}^2}. \qquad (§ 274.)$$

That is, $$K - k = \frac{k}{\overline{OF}^2} \times \overline{A'F}^2.$$

Now let the number of sides of each polygon be indefinitely increased, the two polygons continuing to have the same number of sides.

Then $A'F$, and therefore $K - k$, will approach the limit 0.

But the area of the circle is evidently less than the area of the circumscribed polygon, and greater than the area of the inscribed polygon.

Hence, $K - S$ and $S - k$ will each approach the limit 0.

Therefore, K and k will each approach the limit S.

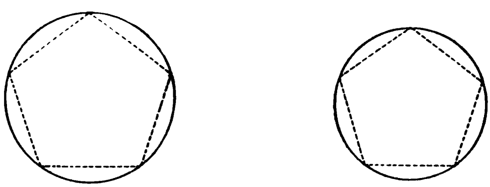
364. Cor. 1. *If a regular polygon be inscribed in a circle, and the number of its sides be indefinitely increased, its apothem approaches the radius of the circle as a limit.*

2. *If a regular polygon be circumscribed about a circle, and the number of its sides be indefinitely increased, its radius approaches the radius of the circle as a limit.*

MEASUREMENT OF THE CIRCLE.

Proposition XII. Theorem.

365. *The circumferences of two circles are to each other as their radii.*



Let C and C' denote the circumferences, and R and R' the radii, of two circles.

To prove $$\frac{C}{C'} = \frac{R}{R'}.$$

Inscribe in the circles regular polygons having the same number of sides, and let P and P' denote their perimeters.

Then, $$\frac{P}{P'} = \frac{R}{R'}. \qquad (\S\ 349.)$$

Whence, $$P \times R' = P' \times R. \qquad (\S\ 231.)$$

Now let the number of sides of each polygon be indefinitely increased, the two polygons continuing to have the same number of sides.

Then, $P \times R'$ will approach the limit $C \times R'$,

and $P' \times R$ will approach the limit $C' \times R$. (§ 363, I.)

By the Theorem of Limits, these limits are equal. (§ 188.)

Whence, $C \times R' = C' \times R$, or $\frac{C}{C'} = \frac{R}{R'}$. (§ 233.)

366. Cor. I. Multiplying the terms of the ratio $\frac{R}{R'}$ by 2, we have

$$\frac{C}{C'} = \frac{2\,R}{2\,R'}.$$

Or, denoting the diameters of the circles by D and D',

$$\frac{C}{C'} = \frac{D}{D'}.$$

That is, *the circumferences of two circles are to each other as their diameters.*

367. Cor. II. The proportion

$$\frac{C}{C'} = \frac{D}{D'} \text{ may be written } \frac{C}{D} = \frac{C'}{D'}. \qquad (\S\ 234.)$$

That is, *the ratio of the circumference of a circle to its diameter has the same value for every circle.*

This constant value is denoted by the symbol π.

That is, $$\frac{C}{D} = \pi. \qquad (1)$$

It is shown by the methods of higher mathematics that the ratio π is incommensurable; hence, its numerical value can only be obtained approximately.

Its value to the nearest fourth decimal place is 3.1416.

368. Cor. III. The equation (1) of § 367 gives

$$C = \pi D.$$

That is, *the circumference of a circle is equal to its diameter multiplied by* π.

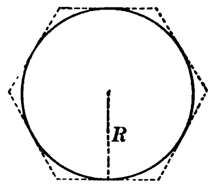
We also have $C = 2\,\pi R.$

That is, *the circumference of a circle is equal to its radius multiplied by* $2\,\pi$.

369. Def. In circles of different radii, *similar arcs, similar segments,* and *similar sectors* are those which correspond to equal central angles.

Proposition XIII. Theorem.

370. *The area of a circle is equal to one-half the product of its circumference and radius.*



Let R denote the radius, C the circumference, and S the area, of a circle.

To prove $\qquad S = \frac{1}{2} C \times R.$

Circumscribe about the circle a regular polygon.

Let P denote its perimeter, and K its area.

Then since the apothem of the polygon is R, we have

$$K = \tfrac{1}{2} P \times R. \qquad (§ 351.)$$

Now let the number of sides of the polygon be indefinitely increased.

Then, $\qquad K$ will approach the limit S,

and $\qquad \frac{1}{2} P \times R$ will approach the limit $\frac{1}{2} C \times R$. (§ 363.)

By the Theorem of Limits, these limits are equal. (§ 188.)

Whence, $\qquad S = \frac{1}{2} C \times R.$

371. Cor. I. We have $C = 2\,\pi R$. (§ 368.)

Whence, $\qquad S = 2\,\pi\,R \times \frac{1}{2}\,R = \pi\,R^2.$

That is, *the area of a circle is equal to the square of its radius multiplied by* π.

Again, $\qquad S = \frac{1}{4}\,\pi \times 4\,R^2 = \frac{1}{4}\,\pi \times (2\,R)^2.$

Or, denoting the diameter of the circle by D,

$$S = \tfrac{1}{4}\,\pi D^2.$$

That is, *the area of a circle is equal to the square of its diameter multiplied by* $\frac{1}{4}\pi$.

372. Cor. II. Let S and S' denote the areas of two circles, R and R' their radii, and D and D' their diameters.

Then, $$\frac{S}{S'} = \frac{\pi R^2}{\pi R'^2} = \frac{R^2}{R'^2},$$

and $$\frac{S}{S'} = \frac{\frac{1}{4}\pi D^2}{\frac{1}{4}\pi D'^2} = \frac{D^2}{D'^2}. \qquad (\S\ 371.)$$

That is, *the areas of two circles are to each other as the squares of their radii, or as the squares of their diameters.*

373. Cor. III. A sector is the same part of the circle that its arc is of the circumference.

Hence, denoting the area and arc of the sector by s and c, and the area and circumference of the circle by S and C, we have $$\frac{s}{S} = \frac{c}{C}, \text{ or } s = c \times \frac{S}{C}.$$

But, $$\frac{S}{C} = \tfrac{1}{2} R. \qquad (\S\ 370.)$$

Whence, $$s = \tfrac{1}{2} c \times R.$$

That is, *the area of a sector is equal to one-half the product of its arc and radius.*

374. Cor. IV. Since similar sectors are like parts of the circles to which they belong (§ 369), it follows that

Similar sectors are to each other as the squares of their radii.

EXERCISES.

28. The area of a circle is equal to four times the area of the circle described upon its radius as a diameter.

29. The area of one circle is $2\frac{7}{9}$ times the area of another. If the radius of the first is 15, what is the radius of the second?

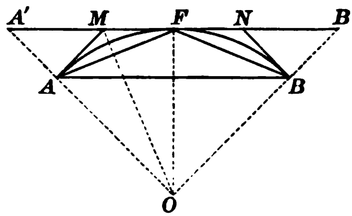
30. The diameters of two circles are 64 and 88, respectively. What is the ratio of their areas?

31. The radii of three circles are 3, 4, and 12, respectively. What is the radius of a circle equivalent to their sum?

32. Find the radius of a circle whose area is one-half the area of a circle whose radius is 9.

PROPOSITION XIV. PROBLEM.

375. *Given p and P, the perimeters of a regular inscribed and of a regular circumscribed polygon of the same number of sides, to find p' and P', the perimeters of the regular inscribed and circumscribed polygons having double the number of sides.*



Let AB be a side of the polygon whose perimeter is p.

Draw the radius OF to the middle point of the arc AB; also, draw OA and OB cutting the tangent at F at A' and B'.

Then, $A'B'$ is a side of the polygon whose perimeter is P. (§§ 346, 347.)

Draw AF and BF; also, draw AM and BN tangent to the circle at A and B, meeting $A'B'$ at M and N.

Then AF and MN are sides of the polygons whose perimeters are p' and P'. (§ 345.)

Hence, if n denotes the number of sides of the polygons whose perimeters are p and P, and therefore $2n$ the number of sides of the polygons whose perimeters are p' and P', we have

$$AB = \frac{p}{n},\ A'B' = \frac{P}{n},\ AF = \frac{p'}{2n},\ \text{and}\ MN = \frac{P'}{2n}. \quad (1)$$

Draw OM; then OM bisects $\angle A'OF$. (§ 175.)

Whence, $$\frac{A'M}{MF} = \frac{OA'}{OF}. \quad (\S\ 249.)$$

But OA' and OF are the radii of the polygons whose perimeters are P and p.

Whence, $\frac{P}{p} = \frac{OA'}{OF}$. (§ 349.)

Then, $\frac{P}{p} = \frac{A'M}{MF}$.

Therefore, $\frac{P+p}{p} = \frac{A'M + MF}{MF}$ (§ 236.)

$$= \frac{A'F}{MF} = \frac{\frac{1}{2}A'B'}{\frac{1}{2}MN}.$$

Then by (1), $\frac{P+p}{p} = \frac{\frac{P}{2n}}{\frac{P'}{4n}} = \frac{2P}{P'}$.

Clearing of fractions,

$$P'(P+p) = 2P \times p.$$

Whence, $P' = \frac{2P \times p}{P+p}$. (2)

Again, in the isosceles triangles ABF and AFM,

$\angle ABF = \angle AFM$. (§§ 193, 197.)

Therefore, ABF and AFM are similar. (§ 256.)

Whence, $\frac{AF}{AB} = \frac{MF}{AF}$, or $\overline{AF}^2 = AB \times MF$.

Then by (1), $\frac{p'^2}{4n^2} = \frac{p}{n} \times \frac{P'}{4n} = \frac{p \times P'}{4n^2}$.

Or, $p'^2 = p \times P'$.

Therefore, $p' = \sqrt{p \times P'}$. (3)

Proposition XV. Problem.

376. *To compute an approximate value of* π (§ 367).

If the diameter of a circle is 1, the side of an inscribed square is $\frac{1}{2}\sqrt{2}$ (§ 353); hence, its perimeter is $2\sqrt{2}$.

Again, the side of a circumscribed square is equal to the diameter; hence, its perimeter is 4.

We then put in formulæ (2) and (3), Prop. XIV.,

$$P = 4, \text{ and } p = 2\sqrt{2} = 2.82843.$$

Whence, $$P' = \frac{2P \times p}{P + p} = 3.31371,$$

and $$p' = \sqrt{p \times P'} = 3.06147;$$

which are the perimeters of the regular circumscribed and inscribed octagons.

Repeating the operation with these values, we put

$$P = 3.31371, \text{ and } p = 3.06147.$$

Whence, $$P' = \frac{2P \times p}{P + p} = 3.18260,$$

and $$p' = \sqrt{p \times P'} = 3.12145;$$

which are the perimeters of the regular circumscribed and inscribed polygons of sixteen sides.

In this way, we form the following table:

No. of Sides.	Perimeter of Circ. Polygon.	Perimeter of Insc. Polygon.
4	4.	2.82843
8	3.31371	3.06147
16	3.18260	3.12145
32	3.15172	3.13655
64	3.14412	3.14033
128	3.14222	3.14128
256	3.14175	3.14151
512	3.14163	3.14157

The last result shows that the circumference of a circle whose diameter is 1 is greater than 3.14157, and less than 3.14163.

Hence, an approximate value of π is 3.1416, correct to the fourth decimal place.

Ex. 33. The diagonals AC, BD, CE, DF, EA, and FB, of a regular hexagon $ABCDEF$, form a regular hexagon whose area is equal to one-third the area of $ABCDEF$.

EXERCISES.

34. Find the circumference and area of a circle whose diameter is 5.

35. Find the radius and area of a circle whose circumference is 75.3984.

36. Find the diameter and circumference of a circle whose area is 201.0624.

If r represents the radius, a the apothem, s the side, and k the area, prove that

37. In a regular octagon,

$$a = \tfrac{1}{2} r \sqrt{2 + \sqrt{2}},\ s = r \sqrt{2 - \sqrt{2}},\text{ and } k = 2 r^2 \sqrt{2}.\ (\S\,375.)$$

38. In a regular dodecagon,

$$a = \tfrac{1}{2} r \sqrt{2 + \sqrt{3}},\ s = r \sqrt{2 - \sqrt{3}},\text{ and } k = 3 r^2.$$

39. In a regular octagon,

$$r = a \sqrt{4 - 2\sqrt{2}},\ s = 2a(\sqrt{2} - 1),\text{ and } k = 8a^2(\sqrt{2} - 1).$$

40. In a regular dodecagon,

$$r = 2a \sqrt{2 - \sqrt{3}},\ s = 2a(2 - \sqrt{3}),\text{ and } k = 12a^2(2 - \sqrt{3}).$$

41. In a regular decagon, $a = \frac{1}{4} r \sqrt{10 + 2\sqrt{5}}$.

42. Given one side of a regular pentagon, to construct the pentagon.

43. Given one side of a regular hexagon, to construct the hexagon.

44. In a given square, to inscribe a regular octagon.

45. In a given equilateral triangle to inscribe a regular hexagon.

46. In a given sector whose central angle is a right angle, to inscribe a square.

47. The area of the square inscribed in a sector whose central angle is a right angle is equal to one-half the square of the radius.

48. The square inscribed in a semicircle is equivalent to two-fifths of the square inscribed in the entire circle.

49. If the diameter of a circle is 48, what is the length of an arc of 85°?

50. If the radius of a circle is $3\sqrt{3}$, what is the area of a sector whose central angle is 152°?

51. If the radius of a circle is 4, what is the area of a segment whose arc is 120°?

52. Find the area of the circle inscribed in a square whose area is 13.

53. Find the area of the square inscribed in a circle whose area is 113.0976.

54. If the apothem of a regular hexagon is 6, what is the area of its circumscribed circle?

55. If the length of a quadrant is 1, what is the diameter of the circle?

56. The length of the arc subtended by a side of a regular inscribed dodecagon is 4.1888. What is the area of the circle?

57. The perimeter of a regular hexagon circumscribed about a circle is $12\sqrt{3}$. What is the circumference of the circle?

58. The area of a regular hexagon inscribed in a circle is $24\sqrt{3}$. What is the area of the circle?

59. The side of an equilateral triangle is 6. Find the areas of its inscribed and circumscribed circles.

60. The side of a square is 8. Find the circumferences of its inscribed and circumscribed circles.

61. Find the area of a segment having for its chord a side of a regular inscribed hexagon, if the radius of the circle is 10.

62. A circular grass-plot, 100 ft. in diameter, is surrounded by a walk 4 ft. wide. Find the area of the walk.

63. Two plots of ground, one a square and the other a circle, contain each 70686 sq. ft. How much longer is the perimeter of the square than the circumference of the circle?

64. A wheel revolves 55 times in travelling 820.743 ft. What is its diameter in inches?

65. What is the number of degrees in an arc whose length is equal to that of the radius of the circle?

66. Find the side of a square equivalent to a circle whose diameter is 3.

67. Find the radius of a circle equivalent to a square whose side is 10.

68. If a circle be circumscribed about a right triangle, and on each of its legs as a diameter a semicircle be described exterior to the triangle, the sum of the areas of the crescents thus formed is equal to the area of the triangle.

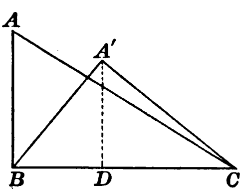
NOTE. For additional exercises on Book V., see p. 228.

APPENDIX TO PLANE GEOMETRY.

MAXIMA AND MINIMA OF PLANE FIGURES.

PROPOSITION I. THEOREM.

377. *Of all triangles formed with two given sides, that in which these sides are perpendicular is the maximum.*



In the triangles ABC and $A'BC$, let $AB = A'B$; and let AB be perpendicular to BC.

To prove $\quad$ area $ABC >$ area $A'BC$.

Draw $A'D$ perpendicular to BC.

Then, $\quad A'B > A'D$. $\quad$ (§ 45.)

That is, $\quad AB > A'D$. $\quad$ (1)

Multiplying both members of (1) by $\frac{1}{2}BC$, we have

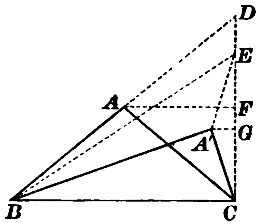
$$\tfrac{1}{2}BC \times AB > \tfrac{1}{2}BC \times A'D.$$

Whence, $\quad$ area $ABC >$ area $A'BC$. $\quad$ (§ 313.)

378. Def. Two figures are said to be *isoperimetric* when they have equal perimeters.

Proposition II. Theorem.

379. *Of isoperimetric triangles having the same base, that which is isosceles is the maximum.*



Let ABC and $A'BC$ be isoperimetric triangles having the same base BC; and let the triangle ABC be isosceles.

To prove area $ABC >$ area $A'BC$.

Produce BA to D, making $AD = AB$, and draw CD.

Then, $\angle BCD$ is a right angle; for it can be inscribed in a semicircle whose centre is A, and radius AB. (§ 196.)

Draw AF and $A'G$ perpendicular to CD.

Take the point E on CD so that $A'E = A'C$, and draw BE.

Then since the triangles ABC and $A'BC$ are isoperimetric,

$$AB + AC = A'B + A'C = A'B + A'E.$$

Whence, $A'B + A'E = AB + AD = BD$.

But, $A'B + A'E > BE$. (Ax. 6.)

Whence, $BD > BE$.

Therefore, $CD > CE$. (§ 51.)

Now AF and $A'G$ are the perpendiculars from the vertices to the bases of the isosceles triangles ACD and $A'CE$.

Whence, $CF = \frac{1}{2} CD$,

and $CG = \frac{1}{2} CE$. (§ 92.)

Therefore, $CF > CG$. (1)

Multiplying both members of (1) by $\frac{1}{2} BC$, we have

$$\tfrac{1}{2} BC \times CF > \tfrac{1}{2} BC \times CG.$$

Whence, area $ABC >$ area $A'BC$. (§ 313.)

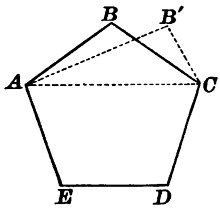
380. Cor. *Of isoperimetric triangles, that which is equilateral is the maximum.*

For if the maximum triangle is not isosceles when any side is taken as the base, its area can be increased by making it isosceles. (§ 379.)

Therefore, the maximum triangle is equilateral.

Proposition III. Theorem.

381. *Of isoperimetric polygons having the same number of sides, that which is equilateral is the maximum.*



Let $ABCDE$ be the maximum of polygons having the given perimeter and the given number of sides.

To prove that $ABCDE$ is equilateral.

If possible, let the sides AB and BC be unequal.

Let $AB'C$ be an isosceles triangle with the base AC, having its perimeter equal to that of the triangle ABC.

Then, area $AB'C >$ area ABC. (§ 379.)

Adding area $ACDE$ to both members, we have

area $AB'CDE >$ area $ABCDE$.

But by hypothesis, $ABCDE$ is the maximum of polygons having the given perimeter.

Therefore, area $AB'CDE$ cannot exceed area $ABCDE$, and hence AB and BC cannot be unequal.

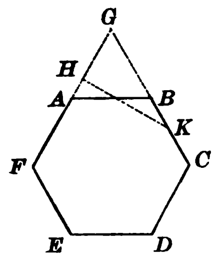
In like manner, we may prove

$$BC = CD = DE, \text{ etc.}$$

Whence, $ABCDE$ is equilateral.

Proposition IV. Theorem.

382. *Of isoperimetric equilateral polygons having the same number of sides, that which is equiangular is the maximum.*



Let A–F be the maximum of equilateral polygons having the same perimeter and the same number of sides.

To prove that A–F is equiangular.

If possible, let $\angle FAB$ be greater than $\angle ABC$.

Produce FA and CB to meet at G.

Then, since $\angle GAB < \angle GBA$, $GA > GB$. (§ 97.)

Lay off $GH = GB$, and $GK = GA$, and draw HK.

Then, $\triangle GHK = \triangle GAB$. (§ 63.)

Taking away the triangle GHK from the entire figure, there remains the polygon $HKCDEF$; and taking away the triangle GAB, there remains the polygon $ABCDEF$.

Hence, $HKCDEF \Leftrightarrow ABCDEF$.

Again, from the equal triangles GHK and GAB, we have

$$HK = AB. \qquad (1)$$

And since $GA = GK$, and $GH = GB$, we have

$$GA - GH = GK - GB, \text{ or } AH = BK.$$

Whence,

$$FH + CK = AF + AH + BC - BK$$
$$= AF + BC. \qquad (2)$$

From (1) and (2), $HKCDEF$ and $ABCDEF$ are isoperimetric.

Then, $HKCDEF$, being equivalent to $ABCDEF$, is the maximum of polygons having the given perimeter.

Therefore, $HKCDEF$ is equilateral. (§ 381.)

But this is impossible, since FH is greater than CK.

Hence, $\angle FAB$ cannot be greater than $\angle ABC$.

Similarly, $\angle FAB$ cannot be less than $\angle ABC$.

Therefore, $\angle FAB = \angle ABC$; and in like manner,

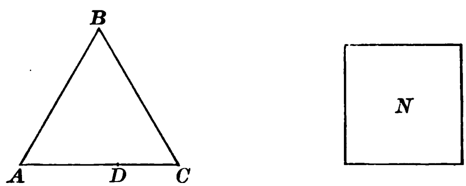
$$\angle ABC = \angle BCD = \angle CDE, \text{ etc.}$$

Whence, A–F is equiangular.

383. Cor. *Of isoperimetric polygons having the same number of sides, that which is regular is the maximum.*

Proposition V. Theorem.

384. *Of two isoperimetric regular polygons, that which has the greater number of sides has the greater area.*



Let ABC be an equilateral triangle, and N an isoperimetric square.

To prove area $N >$ area ABC.

Let D be any point in the side AC of the triangle.

Then the triangle ABC may be regarded as a quadrilateral having the four sides AB, BC, CD, and DA.

Whence, area $N >$ area ABC. (§ 383.)

In like manner, we may prove that the area of a regular pentagon is greater than that of an isoperimetric square; etc.

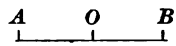
385. Cor. *The area of a circle is greater than the area of any polygon having an equal perimeter.*

SYMMETRICAL FIGURES.

DEFINITIONS.

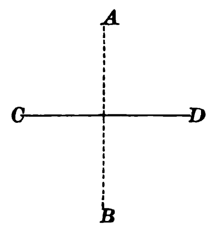
386. Two points are said to be *symmetrical* with respect to a third, called the *centre of symmetry*, when the latter bisects the straight line which joins them.

Thus, if O is the middle point of the straight line AB, the points A and B are symmetrical with respect to O as a centre.



387. The distance of a point from the centre of symmetry is called the *radius of symmetry.*

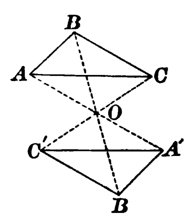
388. Two points are said to be *symmetrical* with respect to a straight line, called the *axis of symmetry,* when the latter bisects at right angles the straight line which joins them.



Thus, if the line CD bisects AB at right angles, the points A and B are symmetrical with respect to CD as an axis.

389. Two geometrical figures are said to be symmetrical with respect to a centre, or with respect to an axis, when to every point of one there corresponds a symmetrical point in the other.

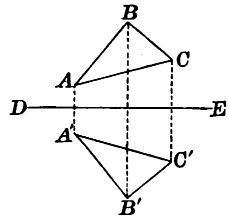
In two such figures, the corresponding parts are called *homologous.*



Thus, if to every point of the triangle ABC there corresponds a symmetrical point of the triangle $A'B'C'$, with respect to the centre O, the triangle $A'B'C'$ is symmetrical to ABC with respect to the centre O.

In this case, the homologous sides are AB and $A'B'$, BC and $B'C'$, and CA and $C'A'$.

Again, if to every point of the triangle ABC there corresponds a symmetrical point of the triangle $A'B'C'$, with respect to the axis DE, the triangle $A'B'C'$ is symmetrical to ABC with respect to the axis DE.



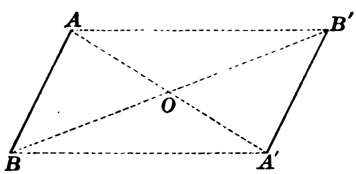
390. A figure is said to be symmetrical with respect to a centre when every straight line drawn through the centre cuts the figure in two points which are symmetrical with respect to that centre.

391. A figure is said to be symmetrical with respect to an axis when it divides it into two figures which are symmetrical with respect to that axis.

Thus, a circumference is symmetrical with respect to its centre as a centre, or with respect to any diameter as an axis.

PROPOSITION VI. THEOREM.

392. *Two straight lines which are symmetrical with respect to a centre are equal and parallel.*



Let the straight lines AB and $A'B'$ be symmetrical with respect to the centre O.

To prove that AB and $A'B'$ are equal and parallel.

Draw AA', BB', AB', and $A'B$.

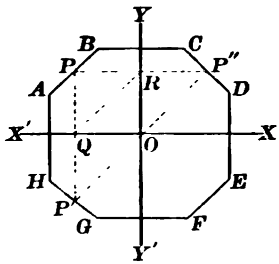
Then, O bisects AA' and BB'. (§ 386.)

Therefore, $AB'A'B$ is a parallelogram. (§ 111.)

Whence, AB and $A'B'$ are equal and parallel. (§ 104.)

PROPOSITION VII. THEOREM.

393. ***If a figure is symmetrical with respect to two axes at right angles to each other, it is symmetrical with respect to their intersection as a centre.***



Let the figure $A\text{–}H$ be symmetrical with respect to the axes XX' and YY', intersecting each other at right angles at O.

To prove that $A\text{–}H$ is symmetrical with respect to O as a centre.

Let P be any point in the perimeter of $A\text{–}H$.

Draw PQ and PR perpendicular to XX' and YY'.

Produce PQ and PR to meet the perimeter of $A\text{–}H$ at P' and P'', and draw QR, OP', and OP''.

Then, since $A\text{–}H$ is symmetrical with respect to XX',

$$PQ = P'Q. \qquad (§ 388.)$$

But $PQ = OR$, and hence OR is equal and parallel to $P'Q$.

Therefore, $OP'QR$ is a parallelogram. (§ 109.)

Whence, QR is equal and parallel to OP'. (§ 104.)

In like manner, we may prove $OP''RQ$ a parallelogram; and therefore QR is equal and parallel to OP''.

Hence, since both OP' and OP'' are equal and parallel to QR, $P'OP''$ is a straight line which is bisected at O.

That is, every straight line drawn through O is bisected at that point; whence, $A\text{–}H$ is symmetrical with respect to O as a centre. (§ 390.)

ADDITIONAL EXERCISES.

BOOK I.

1. The bisectors of the exterior angles of a triangle form a triangle whose angles are respectively the half-sums of the angles of the given triangle taken two and two.

2. If CD is the perpendicular from C to the side AB of the triangle ABC, and CE is the bisector of the angle C, prove that $\angle DCE$ is one-half the difference of the angles A and B.

3. The lines joining the middle points of the adjacent sides of a quadrilateral form a parallelogram whose perimeter is equal to the sum of the diagonals of the quadrilateral.

4. The lines joining the middle points of the opposite sides of a quadrilateral bisect each other.

5. The lines joining the middle points of the opposite sides of a quadrilateral bisect the line joining the middle points of the diagonals.

6. The line joining the middle points of the diagonals of a trapezoid is parallel to the bases and equal to one-half their difference.

7. If D is any point in the side AC of the triangle ABC, and E, F, G, and H are the middle points of AD, CD, BC, and AB respectively, prove that $EFGH$ is a parallelogram.

8. If E and G are the middle points of the sides AB and CD of the quadrilateral $ABCD$, and F and H the middle points of the diagonals AC and BD, prove that $EFGH$ is a parallelogram.

9. If D and E are the middle points of the sides BC and AC of the triangle ABC, and AD be produced to F and BE to G making $DF = AD$ and $EG = BE$, prove that the line FG passes through C.

10. If D is the middle point of the side BC of the triangle ABC, prove $AD < \frac{1}{2}(AB + AC)$.

11. The sum of the medians of a triangle is less than the perimeter, and greater than the semi-perimeter of the triangle. (Ex. 113, p. 69.)

12. If the bisectors of the interior angle at C and the exterior angle at B of the triangle ABC meet at D, prove $\angle BDC = \frac{1}{2}\angle A$.

13. If AD and BD are the bisectors of the exterior angles at the extremities of the hypotenuse of the right triangle ABC, and DE and DF are drawn perpendicular, respectively, to CA and CB produced, prove that $CEDF$ is a square.

14. AD and BE are drawn from two of the vertices of a triangle ABC to the opposite sides, making $\angle BAD = \angle ABE$; if $AD = BE$, prove that the triangle is isosceles.

15. If perpendiculars AE, BF, CG, and DH, be drawn from the vertices of a parallelogram $ABCD$ to any line in its plane, prove that

$$AE + CG = BF + DH.$$

16. If CD is the bisector of the angle C of the triangle ABC, and DF be drawn parallel to AC meeting BC at E and the bisector of the angle exterior to C at F, prove that $DE = EF$.

17. If E and F are the middle points of the sides AB and AC of the triangle ABC, and AD is the perpendicular from A to BC, prove that $\angle EDF = \angle EAF$.

18. If the median drawn from any vertex of a triangle is greater than, equal to, or less than one-half the opposite side, the angle at that vertex is acute, right, or obtuse.

19. Prove that the number of diagonals of a polygon of n sides is

$$\frac{n(n-3)}{2}.$$

20. The sum of the medians of a triangle is greater than three-fourths the perimeter of the triangle.

21. If the lower base AD of a trapezoid $ABCD$ is double the upper base BC, and the diagonals intersect at E, prove that $CE = \frac{1}{3} AC$ and $BE = \frac{1}{3} BD$.

22. If O is the point of intersection of the bisectors of the angles of the equilateral triangle ABC, and OD and OE be drawn respectively perpendicular to BC and parallel to AC, meeting BC at D and E, prove that $DE = \frac{1}{6} BC$.

23. If an equiangular triangle be constructed on each side of a triangle, the lines drawn from their outer vertices to the opposite vertices of the triangle are equal.

24. If two of the medians of a triangle are equal, the triangle is isosceles.

BOOK II.

25. AB and AC are the tangents to a circle from the point A, and D is any point in the smaller of the two arcs subtended by BC. If a tangent to the circle at D meets AB at E and AC at F, prove that the perimeter of the triangle AEF is constant.

26. The line joining the middle points of the arcs subtended by the sides AB and AC of an inscribed triangle ABC cuts AB at F and AC at G. Prove that $AF = AG$.

27. If $ABCD$ is a circumscribed quadrilateral, prove that the angle between the lines joining the opposite points of contact is equal to $\frac{1}{2}(A + C)$.

28. If the sides AB and BC of an inscribed hexagon $ABCDEF$ are parallel to the sides DE and EF respectively, prove that the side AF is parallel to CD.

29. If AB is the common chord of two intersecting circles, and AC and AD are the diameters drawn from A, prove that the line CD passes through B.

30. If AB is a common tangent to two circles which touch each other externally at C, prove that ACB is a right angle.

31. If AB and AC are the tangents to a circle from the point A, and D is any point on the circumference without the triangle ABC, prove that the sum of the angles ABD and ACD is constant.

32. If A, C, B, and D are four points in a straight line, B being between C and D, and EF is a common tangent to the circles described upon AB and CD as diameters, prove that $\angle BAE = \angle DCF$.

33. $ABCD$ is an inscribed quadrilateral, AD being a diameter of the circle. If O is the centre, and the sides AD and BC produced meet at E making $CE = OA$, prove that $\angle AOB = 3 \angle CED$.

34. If $ABCD$ is an inscribed quadrilateral, and its sides AD and BC are produced to meet at P, the tangent at P to the circle circumscribed about the triangle ABP is parallel to CD.

35. $ABCD$ is a quadrilateral inscribed in a circle. If the sides AB and DC produced intersect at E, and the sides AD and BC produced at F, prove that the bisectors of the angles E and F are perpendicular to each other.

36. $ABCD$ is a quadrilateral inscribed in a circle. Another circle is described upon AD as a chord, meeting AB and CD at E and F. Prove that the chords BC and EF are parallel.

37. If $ABCDEFGH$ is an inscribed octagon, the sum of the angles A, C, E, and G is equal to six right angles.

38. If the number of sides of an inscribed polygon is even, the sum of the alternate angles is equal to as many right angles as the polygon has sides less two.

39. If the opposite angles of a quadrilateral are supplementary, the quadrilateral can be inscribed in a circle.

40. The perpendiculars from the vertices of a triangle to the opposite sides are the bisectors of the angles of the triangle formed by joining the feet of the perpendiculars. (Ex. 39.)

CONSTRUCTIONS.

41. Given a side, an adjacent angle, and the radius of the circumscribed circle of a triangle, to construct the triangle.

42. To describe a circle of given radius tangent to a given circle and passing through a given point.

43. Given an angle of a triangle, its bisector, and the length of the perpendicular from its vertex to the opposite side, to construct the triangle.

44. To draw between two given intersecting lines a straight line which shall be equal to one given straight line, and parallel to another.

45. Given an angle of a triangle, and the segments of the opposite side made by the perpendicular from its vertex, to construct the triangle.

46. To draw a parallel to the side BC of the triangle ABC meeting AB and AC in D and E, so that DE may be equal to EC.

47. To draw a parallel to the side BC of the triangle ABC meeting AB and AC in D and E, so that DE may be equal to the sum of BD and CE.

48. Given an angle of a triangle, the perpendicular from the vertex of another angle to the opposite side, and the radius of the circumscribed circle, to construct the triangle.

49. Given the base of a triangle, an adjacent angle, and the sum of the other two sides, to construct the triangle.

50. Given the base of a triangle, an adjacent angle, and the difference of the other two sides, to construct the triangle.

51. Through a given point without a given circle to draw a secant whose internal and external segments shall be equal. (Ex. 67, p. 103.)

52. Given the feet of the perpendiculars from the vertices of a triangle to the opposite sides, to construct the triangle. (Ex. 40.)

BOOK III.

53. State and prove the converse of Prop. XXVI., III.

54. In any triangle, the product of any two sides is equal to the product of the segments of the third side formed by the bisector of the exterior angle at the opposite vertex, minus the square of the bisector. (§ 288.)

55. If the sides of a triangle are $AB = 4$, $AC = 5$, and $BC = 6$, find the length of the bisector of the exterior angle at the vertex A. (§ 250.)

56. ABC is an isosceles triangle. If the perpendicular to AB at A meets the base BC, produced if necessary, at E, and D is the middle point of BE, prove that AB is a mean proportional between BC and BD. (Ex. 88, p. 68.)

57. If D and E, F and G, and H and K are points on the sides AB, BC, and CA, respectively, of the triangle ABC, so taken that $AD = DE = EB$, $BF = FG = GC$, and $CH = HK = KA$, prove that the lines EF, GH, and KD, when produced, form a triangle equal to ABC.

58. If E is the middle point of one of the parallel sides BC of the trapezoid $ABCD$, and AE and DE produced meet DC and AB produced at F and G, prove that FG is parallel to AD.

59. The perpendicular from the intersection of the medians of a triangle to any straight line in the plane of the triangle is one-third the sum of the perpendiculars from the vertices of the triangle to the same line.

60. If E is the middle point of the median AD of the triangle ABC, and BE meets AC at F, prove that $AF = \frac{1}{3} AC$. (§ 140.)

61. The sides AB and BC of the triangle ABC are 3 and 7, respectively, and the length of the bisector of the exterior angle B is $3\sqrt{7}$. Find the side AC.

62. If three or more straight lines divide two parallels proportionally, they pass through a common point.

63. The non-parallel sides of a trapezoid and the line joining the middle points of the parallel sides, if produced, meet in a common point.

64. BD is the perpendicular from the vertex of the right angle to the hypotenuse of the right triangle ABC. If E is any point in AB, and EF be drawn perpendicular to AC, and FG perpendicular to AB, prove that the lines CE and DG are parallel.

65. One segment of a chord drawn through a point 7 units from the centre of a circle is 4 units. If the diameter of the circle is 15 units, what is the other segment?

66. In a right triangle ABC, $\overline{BC}^2 = 3\,\overline{AC}^2$. If CD be drawn from the vertex of the right angle to the middle point of AB, prove that $\angle ACD = 60°$.

67. If D is the middle point of the side BC of the right triangle ABC, and DE be drawn perpendicular to the hypotenuse AB, prove

$$\overline{AE}^2 - \overline{BE}^2 = \overline{AC}^2.$$

68. If BE and CF are the medians drawn from the extremities of the hypotenuse of the right triangle ABC, prove that

$$4\,\overline{BE}^2 + 4\,\overline{CF}^2 = 5\,\overline{BC}^2.$$

69. If ABC and ADC are two angles inscribed in a semicircle, and AE and CF be drawn perpendicular to BD produced, prove

$$\overline{BE}^2 + \overline{BF}^2 = \overline{DE}^2 + \overline{DF}^2.$$

70. If perpendiculars PF, PD, and PE be drawn from any point P to the sides AB, BC, and CA of a triangle, prove that

$$\overline{AF}^2 + \overline{BD}^2 + \overline{CE}^2 = \overline{AE}^2 + \overline{BF}^2 + \overline{CD}^2.$$

71. If BC is the hypotenuse of a right triangle ABC, prove that

$$(AB + BC + CA)^2 = 2\,(AB + BC)\,(BC + CA).$$

72. If any point P be joined to the vertices of the rectangle $ABCD$, prove that $\overline{PA}^2 + \overline{PC}^2 = \overline{PB}^2 + \overline{PD}^2$.

73. If AB and AC are the equal sides of an isosceles triangle, and BD be drawn perpendicular to AC, prove that $2\,AC \times CD = \overline{BC}^2$.

74. If AD and BE are the perpendiculars from the vertices A and B of the acute-angled triangle ABC to the opposite sides, prove

$$AC \times AE + BC \times BD = \overline{AB}^2.$$

75. The sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its four sides. (§ 279.)

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76. To construct a triangle similar to a given triangle, having a given perimeter.

77. To construct a right triangle, having given its perimeter and an acute angle.

78. To describe a circle through two given points, tangent to a given straight line. (§ 282.)

79. If A and B are two points on either side of a given line CD, and AB cuts CD at F, find a point E in CD such that

$$AE : BE = AF : BF.$$

BOOK IV.

80. In the figure on page 176,

(a) Prove that the lines CF and BH are perpendicular.

(b) Prove that the lines AG and BK are parallel.

(c) Prove that the sum of the perpendiculars from H and L to AB produced is equal to AB.

(*d*) Prove that each of the triangles AFH, BEL, and CGK is equivalent to ABC.

(*e*) Prove that C, H, and L are in the same straight line.

(*f*) Prove that the square described upon the sum of AC and BC is equivalent to the square described upon the hypotenuse, plus 4 times the area of ABC.

(*g*) If EL and FH be drawn, prove that the sum of the angles AFH, AHF, BEL, and BLE is equal to a right angle.

(*h*) Prove that $\overline{CF}^2 - \overline{CE}^2 = \overline{AC}^2 - \overline{BC}^2$.

(*i*) If FN and EP are the perpendiculars from F and E to HA and LB produced, prove that the triangles AFN and BEP are each equal to ABC.

(*j*) Prove that $\overline{EL}^2 + \overline{FH}^2 + \overline{GK}^2 = 6\,\overline{AB}^2$.

(*k*) Prove that the lines AL, BH, and CM meet in a common point. (Ex. 80, (*a*).)

(*l*) Prove that HG, LK, and MC when produced meet in a common point.

81. If BE and CF are the medians drawn from the vertices B and C of the triangle ABC, and intersect at D, prove that the triangle BCD is equivalent to the quadrilateral $AEDF$.

82. If D is the middle point of the side BC of a triangle ABC, E the middle point of AD, F of BE, and G of CF, prove that ABC is equivalent to 8 EFG.

83. If E and F are the middle points of the sides AB and CD of a parallelogram $ABCD$, and AF and CE be drawn intersecting BD in H and L, and BF and DE intersecting AC in K and G, prove that $GHKL$ is a parallelogram equivalent to $\frac{1}{6}$ $ABCD$. (§ 140.)

84. Any quadrilateral $ABCD$ is equivalent to a triangle, two of whose sides are equal to the diagonals AC and BD respectively, and include an angle equal to either of the angles between AC and BD.

85. If similar polygons be described upon the sides of a right triangle as homologous sides, the polygon described upon the hypotenuse is equivalent to the sum of the polygons described upon the legs. (§ 323.)

86. If through any point E in the diagonal AC of the parallelogram $ABCD$ parallels to AD and AB be drawn, meeting AB and CD in F and H, and BC and AD in G and K, prove that the triangles EFG and EHK are equivalent.

87. If E, F, G, and H are the middle points of the sides AB, BC, CD, and DA of a square, prove that the lines AF, BG, CH, and DE form a square equivalent to $\frac{1}{5}$ $ABCD$.

88. If E is the intersection of the diagonals AC and BD of a quadrilateral, and the triangles ABE and CDE are equivalent, prove that the sides AD and BC are parallel.

89. If E is any point in the side BC of the parallelogram $ABCD$, and DE be drawn meeting AB produced at F, prove that the triangles ABE and CEF are equivalent.

90. If D is any point in the side AB of a triangle ABC, find a point E in AC such that the triangle ADE is equivalent to one-half the triangle ABC.

91. Find the area of a trapezoid whose parallel sides are 28 and 36, and non-parallel sides 15 and 17, respectively.

BOOK V.

92. The area of the ring included between two concentric circles is equal to the area of the circle whose diameter is that chord of the outer circle which is tangent to the inner.

93. Prove that an equilateral polygon circumscribed about a circle is regular if the number of its sides is odd.

94. Prove that an equiangular polygon inscribed in a circle is regular if the number of its sides is odd.

95. Prove that if the radius of the circle is 1, the side, apothem, and diagonal of a regular inscribed pentagon are

$$\tfrac{1}{2}\sqrt{(10-2\sqrt{5})},\ \tfrac{1}{4}(1+\sqrt{5}),\text{ and }\tfrac{1}{2}\sqrt{(10+2\sqrt{5})}.$$

96. The square of the side of a regular inscribed pentagon, minus the square of the side of a regular inscribed decagon, is equal to the square of the radius.

97. The sum of the perpendiculars drawn to the sides of a regular polygon from any point within the figure is equal to the apothem multiplied by the number of sides of the polygon.

98. In a given equilateral triangle to inscribe three equal circles, tangent to each other and to the sides of the triangle.

99. In a given circle to inscribe three equal circles, tangent to each other and to the given circle.

SOLID GEOMETRY.

Book VI.

LINES AND PLANES IN SPACE. — DIEDRALS. — POLYEDRALS.

394. Def. A plane is said to be *determined* by certain lines or points when one plane, and only one, can be drawn through these lines or points.

Proposition I. Theorem.

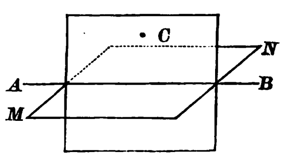
395. *A plane is determined*

I. *By a straight line and a point without the line.*

II. *By three points not in the same straight line.*

III. *By two intersecting straight lines.*

IV. *By two parallel straight lines.*

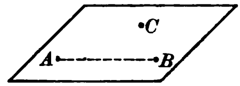


I. Let C be a point without the straight line AB.

To prove that a plane is determined (§ 394) by AB and C.

If any plane, as MN, be drawn through AB, it may be revolved about AB as an axis until it contains the point C.

Hence, one plane, and only one, can be drawn through AB and C.



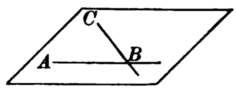
II. Let A, B, and C be three points not in the same straight line.

To prove that a plane is determined by A, B, and C.

Draw AB.

By I., one plane, and only one, can be drawn through the line AB and the point C.

Hence, one plane, and only one, can be drawn through A, B, and C.



III. Let AB and BC be two intersecting straight lines.

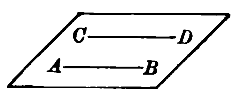
To prove that a plane is determined by AB and BC.

By I., one plane, and only one, can be drawn through AB and any point C of BC.

But since this plane contains the points B and C, it must contain the line BC.

[A plane is a surface such that the straight line joining any two of its points lies entirely in the surface.] (§ 8.)

Hence, one plane, and only one, can be drawn through AB and BC.



IV. Let AB and CD be two parallel lines.

To prove that a plane is determined by AB and CD.

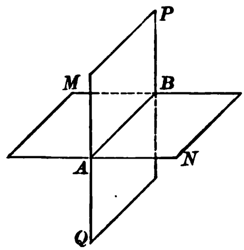
The parallels AB and CD lie in the same plane (§ 52).

And by I., but one plane can be drawn through AB and any point C of CD.

Hence, one plane, and only one, can be drawn through AB and CD.

PROPOSITION II. THEOREM.

396. *The intersection of two planes is a straight line.*



Let the line AB be the intersection of the planes MN and PQ.

To prove AB a straight line.

Let a straight line be drawn between the points A and B.

This line must lie in MN, and also in PQ.

[A plane is a surface such that the straight line joining any two of its points lies entirely in the surface.] (§ 8.)

Then it must be the intersection of MN and PQ.

Whence, AB is a straight line.

397. DEF. If a straight line meets a plane, the point of intersection is called the *foot* of the line.

A straight line is said to be *perpendicular to a plane* when it is perpendicular to every straight line drawn in the plane through its foot.

A straight line is said to be *parallel to a plane* when it cannot meet the plane however far they may be produced.

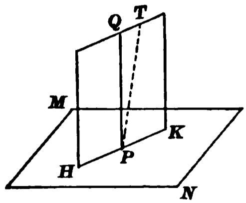
Two planes are said to be *parallel to each other* when they cannot meet however far they may be produced.

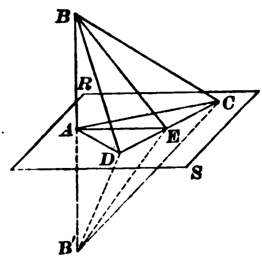
398. SCH. The following form of the second definition of § 397 is given for convenience of reference:

A perpendicular to a plane is perpendicular to every straight line drawn in the plane through its foot.

PROPOSITION III. THEOREM.

399. ***At a given point in a plane, one perpendicular to the plane can be drawn, and but one.***





Let P be the given point in the plane MN.

To prove that a perpendicular can be drawn to MN at P, and but one.

At any point A of the straight line AB draw the lines AC and AD perpendicular to AB.

Let RS be the plane determined by AC and AD.

Let AE be any other straight line drawn through the point A in the plane RS; and draw the line CED intersecting AC, AE, and AD in C, E, and D.

Produce BA to B', making $AB' = AB$.

Draw BC, BE, BD, $B'C$, $B'E$, and $B'D$.

In the triangles BCD and $B'CD$, the side CD is common.

And since AC and AD are perpendicular to BB' at its middle point,

$$BC = B'C, \text{ and } BD = B'D.$$

[If a perpendicular be erected at the middle point of a straight line, any point in the perpendicular is equally distant from the extremities of the line.] (§ 40, I.)

Whence, $\triangle BCD = \triangle B'CD.$

[Two triangles are equal when the three sides of one are equal respectively to the three sides of the other.] (§ 69.)

Now revolve triangle BCD about CD as an axis until it coincides with triangle $B'CD$.

Then B will fall at B', and the line BE will coincide with $B'E$; that is, $BE = B'E$.

Hence, since the points A and E are each equally distant from B and B', AE is perpendicular to BB'.

[Two points, each equally distant from the extremities of a straight line, determine a perpendicular at its middle point.] (§ 43.)

But AE is *any* straight line drawn through A in RS.

Then, AB is perpendicular to *every* straight line drawn through its foot in the plane RS.

Whence, AB is perpendicular to RS.

[A straight line is said to be perpendicular to a plane when it is perpendicular to every straight line drawn in the plane through its foot.] (§ 397.)

Now apply the plane RS to the plane MN so that the point A shall fall at P; and let AB take the position PQ.

Then, PQ will be perpendicular to MN.

Hence, a perpendicular can be drawn to MN at P.

If possible, let PT be another perpendicular to MN at P; and let the plane determined by PQ and PT intersect MN in the line HK.

Then, both PQ and PT are perpendicular to HK.

[A perpendicular to a plane is perpendicular to every straight line drawn in the plane through its foot.] (§ 398.)

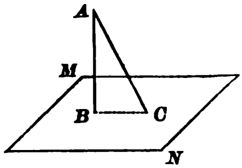
But in the plane HKT, only one perpendicular can be drawn to HK at P.

[At a given point in a straight line, but one perpendicular to the line can be drawn.] (§ 28.)

Hence, but one perpendicular can be drawn to MN at P.

400. COR. I. *A straight line perpendicular to each of two straight lines at their point of intersection is perpendicular to their plane.*

401. COR. II. *From a given point without a plane, one perpendicular to the plane can be drawn, and but one.*



The latter statement is proved as follows:

If possible, let AB and AC be two perpendiculars from A to the plane MN.

Draw BC; then the triangle ABC will have two right angles.

[A perpendicular to a plane is perpendicular to every straight line drawn in the plane through its foot.] (§ 398.)

But this is impossible.

Then but one perpendicular can be drawn from A to MN.

402. Cor. III. *The perpendicular is the shortest line that can be drawn from a point to a plane.*

Let AB be the perpendicular from A to the plane MN, and AC any other straight line from A to MN.

To prove $AB < AC$.

Draw BC; then, since AB is perpendicular to BC,

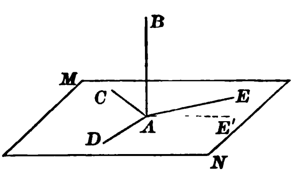
$$AB < AC.$$

[The perpendicular is the shortest line that can be drawn from a point to a straight line.] (§ 45.)

403. Sch. The *distance* of a point from a plane signifies the length of the perpendicular from the point to the plane.

Proposition IV. Theorem.

404. *All the perpendiculars to a straight line at a given point lie in a plane perpendicular to the line.*



Let AC and AD be perpendicular to the line AB at A.

Then the plane MN, determined by AC and AD, is perpendicular to AB.

[A straight line perpendicular to each of two straight lines at their point of intersection is perpendicular to their plane.] (§ 400.)

Let AE be any other perpendicular to AB at A.

To prove that AE lies in MN.

Let the plane determined by AB and AE intersect MN in AE'; then, AB is perpendicular to AE'.

[A perpendicular to a plane is perpendicular to every straight line drawn in the plane through its foot.] (§ 398.)

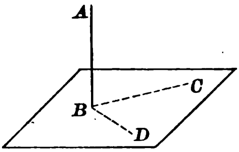
But in the plane ABE, but one perpendicular can be drawn to AB at A.

[At a given point in a straight line, but one perpendicular to the line can be drawn.] (§ 28.)

Then, AE' and AE coincide, and AE lies in the plane MN.

405. Cor. I. *Through a given point in a straight line, a plane can be drawn perpendicular to the line, and but one.*

406. Cor. II. *Through a given point without a straight line, a plane can be drawn perpendicular to the line, and but one.*



Let C be the given point without the straight line AB.

To prove that a plane can be drawn through C perpendicular to AB, and but one.

Draw CB perpendicular to AB, and let BD be any other perpendicular to AB at B.

Then the plane determined by BD and BC will be a plane drawn through C perpendicular to AB.

[A straight line perpendicular to each of two straight lines at their point of intersection is perpendicular to their plane.] (§ 400.)

But only one perpendicular can be drawn from C to AB.

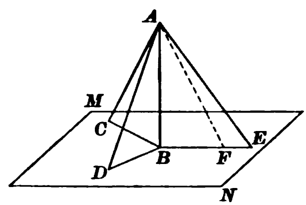
Hence, but one plane can be drawn through C perpendicular to AB.

PROPOSITION V. THEOREM.

407. *If oblique lines be drawn from a point to a plane,*

I. *Two oblique lines cutting off equal distances from the foot of the perpendicular from the point to the plane are equal.*

II. *Of two oblique lines cutting off unequal distances from the foot of the perpendicular, the more remote is the greater.*



I. Let the oblique lines AC and AD meet the plane MN at equal distances from the foot of the perpendicular AB.

To prove $\qquad AC = AD.$

Draw BC and BD.

In the triangles ABC and ABD, the side AB is common.

Also, $\qquad \angle ABC = \angle ABD.$

[A perpendicular to a plane is perpendicular to every straight line drawn in the plane through its foot.] (§ 398.)

And by hypothesis, $\qquad BC = BD.$

Then, $\qquad \triangle ABC = \triangle ABD.$

[Two triangles are equal when two sides and the included angle of one are equal respectively to two sides and the included angle of the other.] (§ 63.)

Whence, $\qquad AC = AD.$

[In equal figures, the homologous parts are equal.] (§ 66.)

II. Let the line AE meet MN at a greater distance from B than AC.

To prove $\qquad AE > AC.$

Draw BE; on BE take $BF = BC$, and draw AF.

Then, $AF = AC.$

[If oblique lines be drawn from a point to a plane, two oblique lines cutting off equal distances from the foot of the perpendicular from the point to the plane are equal.] (§ 407, I.)

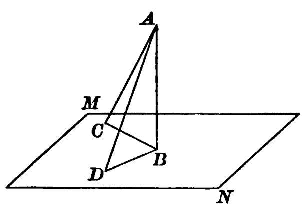
But, $AE > AF.$

[If oblique lines be drawn from a point to a straight line, of two oblique lines cutting off unequal distances from the foot of the perpendicular, the more remote is the greater.] (§ 48, II.)

Whence, $AE > AC.$

Proposition VI. Theorem.

408. (Converse of Prop. V., I.) *Two equal oblique lines from a point to a plane cut off equal distances from the foot of the perpendicular from the point to the plane.*



Let AC and AD be equal oblique lines, and AB the perpendicular, from A to the plane MN; and draw BC and BD.

To prove $BC = BD.$

In the triangles ABC and ABD, AB is common.

And by hypothesis, $AC = AD.$

Also, ABC and ABD are right angles.

[A perpendicular to a plane is perpendicular to every straight line drawn in the plane through its foot.] (§ 398.)

Whence, $\triangle ABC = \triangle ABD.$

[Two right triangles are equal when the hypotenuse and a leg of one are equal respectively to the hypotenuse and a leg of the other.] (§ 88.)

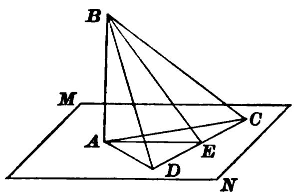
Therefore, $BC = BD.$

409. COR. (Converse of Prop. V., II.) *If two unequal oblique lines be drawn from a point to a plane, the greater cuts off the greater distance from the foot of the perpendicular from the point to the plane.*

(The proof is left to the student.)

PROPOSITION VII. THEOREM.

410. *If through the foot of a perpendicular to a plane a line be drawn at right angles to any line in the plane, the line drawn from its intersection with this line to any point in the perpendicular will be perpendicular to the line in the plane.*



Let AB be perpendicular to the plane MN.

Draw AE perpendicular to any line CD in MN, and join E to any point B in AB.

To prove BE perpendicular to CD.

On CD take $EC = ED$; and draw AC, AD, BC, and BD.

Then, $AC = AD$.

[If a perpendicular be erected at the middle point of a straight line, any point in the perpendicular is equally distant from the extremities of the line.] (§ 40, I.)

Therefore, $BC = BD$.

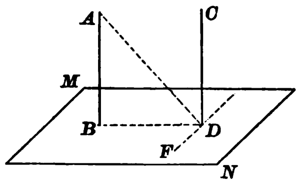
[If oblique lines be drawn from a point to a plane, two oblique lines cutting off equal distances from the foot of the perpendicular from the point to the plane are equal.] (§ 407, I.)

Whence, BE is perpendicular to CD.

[Two points, each equally distant from the extremities of a straight line, determine a perpendicular at its middle point.] (§ 43.)

PROPOSITION VIII. THEOREM.

411. *Two perpendiculars to the same plane are parallel.*



Let the lines AB and CD be perpendicular to the plane MN.

To prove AB and CD parallel.

Let A be any point of AB, and draw AD and BD.

Also, draw DF in the plane MN perpendicular to BD.

Then CD is perpendicular to DF.

[A perpendicular to a plane is perpendicular to every straight line drawn in the plane through its foot.] (§ 398.)

Also, AD is perpendicular to DF.

[If through the foot of a perpendicular to a plane a line be drawn at right angles to any line in the plane, the line drawn from its intersection with this line to any point in the perpendicular will be perpendicular to the line in the plane.] (§ 410.)

Therefore, CD, AD, and BD, being perpendicular to DF at D, lie in the same plane.

[All the perpendiculars to a straight line at a given point lie in a plane perpendicular to the line.] (§ 404.)

Hence, AB and CD lie in the same plane.

[A plane is a surface such that the straight line joining any two of its points lies entirely in the surface.] (§ 8.)

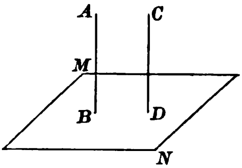
Again, AB and CD are perpendicular to BD.

[A perpendicular to a plane is perpendicular to every straight line drawn in the plane through its foot.] (§ 398.)

Whence, AB and CD are parallel.

[Two perpendiculars to the same straight line are parallel.] (§ 54.)

412. Cor. I. *If one of two parallels is perpendicular to a plane, the other is also perpendicular to the plane.*



Let the lines AB and CD be parallel, and let AB be perpendicular to the plane MN.

To prove CD perpendicular to MN.

A perpendicular from C to MN will be parallel to AB.

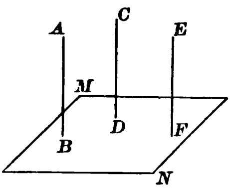
[Two perpendiculars to the same plane are parallel.] (§ 411.)

But through C, only one parallel can be drawn to AB.

[But one straight line can be drawn through a given point parallel to a given straight line.] (§ 53.)

Whence, CD is perpendicular to MN.

413. Cor. II. *If each of two straight lines is parallel to a third, they are parallel to each other.*



Let the lines AB and CD be parallel to EF.

To prove AB and CD parallel.

Draw the plane MN perpendicular to EF.

Then each of the lines AB and CD is perpendicular to MN.

[If one of two parallels is perpendicular to a plane, the other is also perpendicular to the plane.] (§ 412.)

Whence, AB and CD are parallel.

[Two perpendiculars to the same plane are parallel.] (§ 411.)

EXERCISES.

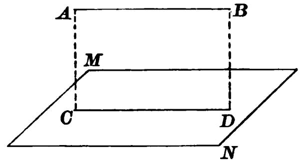
1. What is the locus (§ 141) of the perpendiculars to a given straight line AB at the point A?

2. What is the locus of points equally distant from the circumference of a given circle?

3. If a plane bisects a straight line at right angles, any point in the plane is equally distant from the extremities of the line.

Proposition IX. Theorem.

414. *A straight line parallel to a line in a plane is parallel to the plane.*



Let AB be parallel to the line CD in the plane MN.

To prove AB parallel to MN.

The parallels AB and CD lie in a plane, which intersects MN in the line CD.

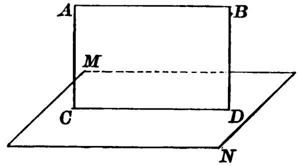
Hence, if AB meets MN, it must be at some point of CD.

But AB, being parallel to CD, cannot meet it.

Then AB and MN cannot meet, and are parallel (§ 397).

Proposition X. Theorem.

415. *If a straight line is parallel to a plane, the intersection of the plane with any plane drawn through the line is parallel to the line.*



Let the line AB be parallel to the plane MN; and let CD be the intersection of MN with a plane drawn through AB.

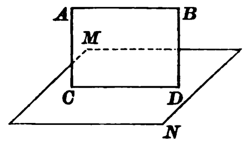
To prove AB and CD parallel.

The lines AB and CD lie in the same plane.

And since AB cannot meet the plane MN however far they may be produced, it cannot meet CD.

Therefore, AB and CD are parallel (§ 52).

416. Cor. *If a line and a plane are parallel, a parallel to the line through any point of the plane lies in the plane.*



Let the line AB be parallel to the plane MN; and through any point C of MN draw CD parallel to AB.

To prove that CD lies in MN.

The plane determined by AB and C intersects MN in a parallel to AB.

[If a straight line is parallel to a plane, the intersection of the plane with any plane drawn through the line is parallel to the line.] (§ 415.)

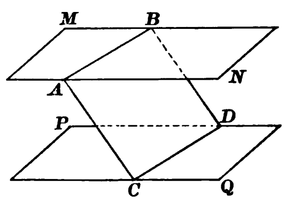
But through C, only one parallel can be drawn to AB.

[But one straight line can be drawn through a given point parallel to a given straight line.] (§ 53.)

Whence, CD lies in MN.

Proposition XI. Theorem.

417. *If two parallel planes are cut by a third plane, the intersections are parallel.*



Let the parallel planes MN and PQ be cut by the plane AD in the lines AB and CD, respectively.

To prove AB and CD parallel.

The lines AB and CD lie in the same plane.

And since the planes MN and PQ cannot meet however far they may be produced, AB and CD cannot meet.

Therefore, AB and CD are parallel (§ 52).

418. COR. *Parallel lines included between parallel planes are equal.*

Let AC and BD be parallel lines included between the parallel planes MN and PQ.

To prove $AC = BD$.

Let the plane determined by AC and BD intersect MN and PQ in the lines AB and CD.

Then, AB and CD are parallel.

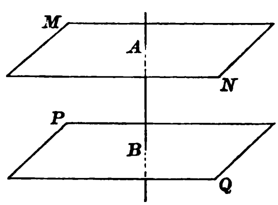
[If two parallel planes are cut by a third plane, the intersections are parallel.] (§ 417.)

Therefore, $AC = BD$.

[Parallel lines included between parallel lines are equal.] (§ 105.)

PROPOSITION XII. THEOREM.

419. *Two planes perpendicular to the same straight line are parallel.*



Let the planes MN and PQ be perpendicular to the line AB.

To prove MN and PQ parallel.

If MN and PQ are not parallel, they will meet if sufficiently produced; let C be a point in their intersection.

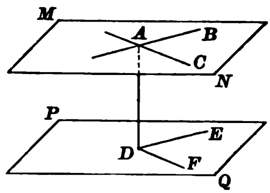
There will then be two planes drawn through C perpendicular to AB, which is impossible.

[Through a given point without a straight line, but one plane can be drawn perpendicular to the line.] (§ 406.)

Whence, MN and PQ cannot meet, and are parallel.

PROPOSITION XIII. THEOREM.

420. *If each of two intersecting lines is parallel to a plane, their plane is parallel to the given plane.*



Let AB and AC be parallel to the plane PQ.

To prove their plane MN parallel to PQ.

Draw AD perpendicular to PQ.

Through D draw DE and DF parallel to AB and AC.

Then, DE and DF lie in the plane PQ.

[If a line and a plane are parallel, a parallel to the line through any point of the plane lies in the plane.] (§ 416.)

Whence, AD is perpendicular to DE and DF.

[A perpendicular to a plane is perpendicular to every straight line drawn in the plane through its foot.] (§ 398.)

Therefore, AD is perpendicular to AB and AC.

[A straight line perpendicular to one of two parallels is perpendicular to the other.] (§ 56.)

Hence, AD is perpendicular to MN.

[A straight line perpendicular to each of two straight lines at their point of intersection is perpendicular to their plane.] (§ 400.)

Then MN and PQ are parallel.

[Two planes perpendicular to the same straight line are parallel.] (§ 419.)

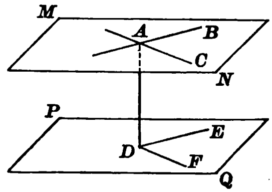
EXERCISES.

4. A line parallel to each of two intersecting planes is parallel to their intersection.

5. A straight line and a plane perpendicular to the same straight line are parallel.

PROPOSITION XIV. THEOREM.

421. *A straight line perpendicular to one of two parallel planes is perpendicular to the other also.*



Let MN and PQ be parallel planes; and let the line AD be perpendicular to PQ.

To prove AD perpendicular to MN.

Pass any two planes through AD, intersecting MN in AB and AC, and PQ in DE and DF.

Then AB is parallel to DE, and AC to DF.

[If two parallel planes are cut by a third plane, the intersections are parallel.] (§ 417.)

But AD is perpendicular to DE and DF.

[A perpendicular to a plane is perpendicular to every straight line drawn in the plane through its foot.] (§ 398.)

Whence, AD is perpendicular to AB and AC.

[A straight line perpendicular to one of two parallels is perpendicular to the other.] (§ 56.)

Therefore, AD is perpendicular to MN.

[A straight line perpendicular to each of two straight lines at their point of intersection is perpendicular to their plane.] (§ 400.)

422. COR. I. *Two parallel planes are everywhere equally distant* (§ 403).

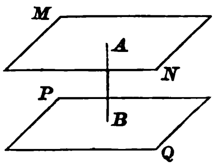
For all common perpendiculars to the planes are parallel.

[Two perpendiculars to the same plane are parallel.] (§ 411.)

Therefore they are all equal.

[Parallel lines included between parallel planes are equal.] (§ 418.)

423. Cor. II. *Through a given point a plane can be drawn parallel to a given plane, and but one.*



Let A be the given point, and PQ the given plane.

To prove that a plane can be drawn through A parallel to PQ, and but one.

Draw AB perpendicular to PQ.

Through A pass the plane MN perpendicular to AB.

Then MN will be parallel to PQ.

[Two planes perpendicular to the same straight line are parallel.] (§ 419.)

If another plane could be drawn through A parallel to PQ, it would be perpendicular to AB.

[A straight line perpendicular to one of two parallel planes is perpendicular to the other also.] (§ 421.)

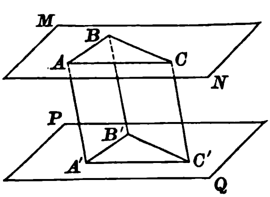
It would then coincide with MN.

[Through a given point in a straight line, but one plane can be drawn perpendicular to the line.] (§ 405.)

Then but one plane can be drawn through A parallel to PQ.

Proposition XV. Theorem.

424. *If two angles not in the same plane have their sides parallel and extending in the same direction, they are equal, and their planes are parallel.*



Let MN and PQ be the planes of the angles BAC and $B'A'C'$; and let AB and AC be parallel respectively to $A'B'$ and $A'C'$, and extend in the same direction.

I. To prove $\angle BAC = \angle B'A'C'$.

Lay off $AB = A'B'$, and $AC = A'C'$; and draw AA', BB', CC', BC, and $B'C'$.

Then since AB is equal and parallel to $A'B'$, the figure $ABB'A'$ is a parallelogram.

[If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.] (§ 109.)

Whence, AA' is equal and parallel to BB'.

[The opposite sides of a parallelogram are equal.] (§ 104.)

In like manner, AA' is equal and parallel to CC'.

Therefore, BB' is equal and parallel to CC'.

[If each of two straight lines is parallel to a third, they are parallel to each other.] (§ 413.)

Whence, $BB'C'C$ is a parallelogram, and $BC = B'C'$.

Therefore, $\triangle ABC = \triangle A'B'C'$.

[Two triangles are equal when the three sides of one are equal respectively to the three sides of the other.] (§ 69.)

Whence, $\angle BAC = \angle B'A'C'$.

[In equal figures, the homologous parts are equal.] (§ 66.)

II. To prove MN parallel to PQ.

The lines AB and AC are each parallel to the plane PQ.

[A straight line parallel to a line in a plane is parallel to the plane.] (§ 414.)

Therefore, MN is parallel to PQ.

[If each of two intersecting lines is parallel to a plane, their plane is parallel to the given plane.] (§ 420.)

EXERCISES.

6. What is the locus of points equally distant from a given plane?

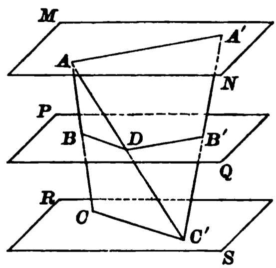
7. If two planes are parallel, a line parallel to one of them through any point of the other lies in the other.

8. If two planes are parallel to a third plane, they are parallel to each other.

9. If a line is parallel to a plane, it is everywhere equally distant from the plane.

Proposition XVI. Theorem.

425. *If two straight lines are cut by three parallel planes, the corresponding segments are proportional.*



Let the parallel planes MN, PQ, and RS intersect the lines AC and $A'C'$ in the points A, B, C, and A', B', C', respectively.

To prove $$\frac{AB}{BC} = \frac{A'B'}{B'C'}.$$

Draw AC'.

Through AC and AC' pass a plane, intersecting PQ and RS in the lines BD and CC'.

Then BD is parallel to CC'.

[If two parallel planes are cut by a third plane, the intersections are parallel.] (§ 417.)

Therefore, $$\frac{AB}{BC} = \frac{AD}{DC'}. \quad (1)$$

[A parallel to one side of a triangle divides the other two sides proportionally.] (§ 245.)

In like manner, $$\frac{AD}{DC'} = \frac{A'B'}{B'C'}. \quad (2)$$

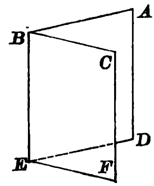
From (1) and (2), $$\frac{AB}{BC} = \frac{A'B'}{B'C'}.$$

Ex. 10. Through a given point a plane can be drawn parallel to any two straight lines in space. (§ 420.)

DIEDRALS.

Definitions.

426. If two planes meet in a straight line, the figure formed is called a *diedral angle*, or simply a *diedral*.



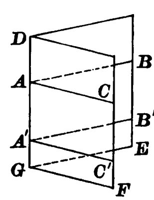
The line of intersection of the planes is called the *edge* of the diedral, and the planes are called its *faces*.

Thus, in the diedral formed by the planes BD and BF, BE is the edge, and BD and BF are the faces.

427. A diedral may be designated by two letters on its edge; or, if several diedrals have a common edge, by four letters, one in each face and two on the edge, the letters on the edge being named between the other two.

Thus, the above diedral may be designated BE, or $ABEC$.

428. The *plane angle* of a diedral is the angle formed by two straight lines drawn one in each face, perpendicular to the edge at the same point.



Thus, if the lines AB and AC be drawn in the faces DE and DF, respectively, perpendicular to DG at A, BAC is the plane angle of the diedral DG.

429. Let the lines $A'B'$ and $A'C'$ be drawn in the faces DE and DF, respectively, perpendicular to DG at A'.

Then, $A'B'$ is parallel to AB, and $A'C'$ to AC. (§ 54.)

Whence, $\angle B'A'C' = \angle BAC$. (§ 424.)

That is, *the plane angle of a diedral is of the same magnitude at whatever point of the edge it may be drawn.*

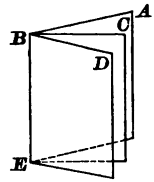
430. Two diedrals are *equal* when their faces may be made to coincide.

431. It is evident that *two diedrals are equal when their plane angles are equal.*

432. Conversely, *the plane angles of equal diedrals are equal.*

433. A plane perpendicular to the edge of a diedral intersects the faces in lines perpendicular to the edge (§ 398).

Hence, *a plane perpendicular to the edge of a diedral intersects the faces in lines which form the plane angle of the diedral.*

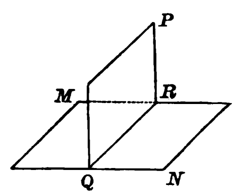


434. Two diedrals are said to be *adjacent* when they have the same edge, and a common face between them; as *ABEC* and *CBED.*

Two diedrals are said to be *vertical* when the faces of one are the extensions of the faces of the other.

435. Through a given straight line in a plane, a plane may be drawn meeting the given plane in such a way as to make the adjacent diedrals equal. (Compare § 27.)

Each of the equal diedrals is called a *right diedral,* and the planes are said to be *perpendicular* to each other.



Thus, if the plane *PQ* be drawn meeting the plane *MN* in such a way as to make the adjacent diedrals *PRQM* and *PRQN* equal, each of these is a right diedral, and *MN* and *PQ* are perpendicular to each other.

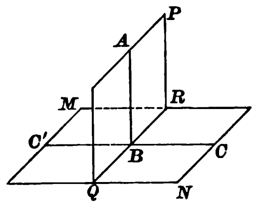
436. Through a given line in a plane but one plane can be drawn perpendicular to the given plane. (Compare § 28.)

437. The *projection of a point on a plane* is the foot of the perpendicular drawn from the point to the plane.

The *projection of a line on a plane* is the locus (§ 141) of the projections of its points.

Proposition XVII. Theorem.

438. *The plane angle of a right diedral is a right angle.*



Let the planes PQ and MN be perpendicular to each other, and intersect in the line QR.

Let ABC and ABC' be the plane angles of the diedrals $PRQN$ and $PRQM$.

To prove ABC a right angle.

Since PQ is perpendicular to MN, we have

diedral $PRQN$ = diedral $PRQM$. (§ 435.)

Whence, $\angle ABC = \angle ABC'$. (§ 432.)

Therefore, ABC is a right angle. (§ 27.)

439. Cor. (Converse of Prop. XVII.) *If the plane angle of a diedral is a right angle, the faces of the diedral are perpendicular to each other.*

Let the planes PQ and MN intersect in the line QR.

Let ABC and ABC' be the plane angles of the diedrals $PRQN$ and $PRQM$, and let ABC be a right angle.

To prove PQ perpendicular to MN.

Since ABC is a right angle, we have

$\angle ABC = \angle ABC'$. (§ 27.)

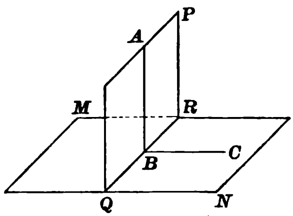
Whence, diedral $PRQN$ = diedral $PRQM$. (§ 431.)

Therefore, PQ is perpendicular to MN. (§ 435.)

Ex. 11. Through any given straight line a plane can be drawn parallel to any other straight line. (§ 414.)

PROPOSITION XVIII. THEOREM.

440. *If two planes are perpendicular to each other, a straight line drawn in one of them perpendicular to their intersection is perpendicular to the other.*



Let the plane PQ be perpendicular to MN.

Let QR be their intersection, and draw AB in the plane PQ perpendicular to QR.

To prove AB perpendicular to MN.

Draw BC in the plane MN perpendicular to QR.

Then ABC is the plane angle of the diedral $PRQN$. (§ 428.)

Whence, ABC is a right angle. (§ 438.)

Therefore AB, being perpendicular to BC and BQ at B, is perpendicular to the plane MN. (§ 400.)

441. COR. I. *If two planes are perpendicular, a perpendicular to one of them at any point of their intersection lies in the other.*

Let the plane PQ be perpendicular to MN; at any point B in their intersection QR, draw AB perpendicular to MN.

To prove that AB lies in PQ.

A line drawn in PQ perpendicular to QR at B will be perpendicular to MN. (§ 440.)

But at the point B, but one perpendicular can be drawn to MN. (§ 399.)

Hence, AB lies in PQ.

442. COR. II. *If two planes are perpendicular, a perpendicular to one from any point of the other lies in the other.*

Let the plane PQ be perpendicular to MN; and through any point A of PQ draw AB perpendicular to MN.

To prove that AB lies in PQ.

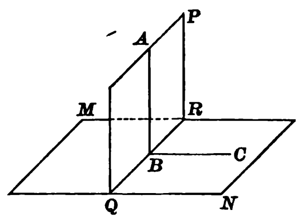
A line drawn in PQ through the point A, perpendicular to the intersection QR, will be perpendicular to MN. (§ 440.)

But from the point A, but one perpendicular can be drawn to MN. (§ 401.)

Hence, AB lies in PQ.

PROPOSITION XIX. THEOREM.

443. *If a straight line is perpendicular to a plane, every plane drawn through the line is perpendicular to the plane.*



Let the line AB be perpendicular to the plane MN; and let PQ be any plane drawn through AB.

To prove PQ perpendicular to MN.

Let QR be the intersection of PQ and MN, and draw BC in the plane MN perpendicular to QR.

Now AB is perpendicular to BQ. (§ 398.)

Then ABC is the plane angle of the diedral $PRQN$. (§ 428.)

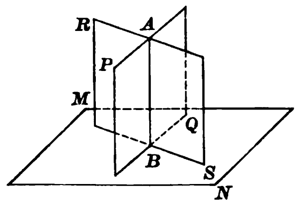
But ABC is a right angle. (§ 398.)

Hence, PQ is perpendicular to MN. (§ 439.)

444. COR. *A plane perpendicular to the edge of a diedral is perpendicular to its faces.*

PROPOSITION XX. THEOREM.

445. *A plane perpendicular to each of two intersecting planes is perpendicular to their intersection.*

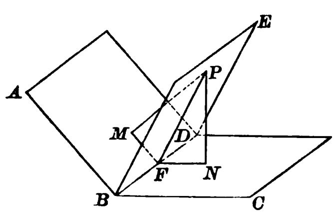


Let the planes PQ and RS be perpendicular to MN.
To prove their intersection AB perpendicular to MN.

Let a perpendicular be drawn to MN at B.
This perpendicular will lie in both PQ and RS. (§ 441.)
It must therefore be their line of intersection.
Hence, AB is perpendicular to MN.

PROPOSITION XXI. THEOREM.

446. *Every point in the bisecting plane of a diedral is equally distant from the faces of the diedral.*



From any point P in the bisecting plane BE of the diedral $ABDC$, draw PM and PN perpendicular to AD and CD.

To prove $PM = PN$.

Let the plane determined by PM and PN intersect the planes AD, BE, and CD in FM, FP, and FN.

The plane $PMFN$ is perpendicular to the planes AD and CD. (§ 443.)

Then the plane $PMFN$ is perpendicular to BD. (§ 445.)

Therefore, PFM and PFN are the plane angles of the diedrals $ABDE$ and $CBDE$. (§ 433.)

Whence, $\angle PFM = \angle PFN$. (§ 432.)

Now in the right triangles PFM and PFN, PF is common.

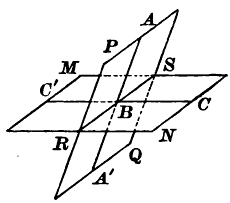
Also, $\angle PFM = \angle PFN$.

Therefore, $\triangle PFM = \triangle PFN$. (§ 70.)

Whence, $PM = PN$. (§ 66.)

Proposition XXII. Theorem.

447. *If two planes intersect, the vertical diedrals are equal.*



Let the planes MN and PQ intersect in the line RS.

To prove diedral $PRSN$ = diedral $MRSQ$.

Let ABC and $A'BC'$ be the plane angles of the diedrals $PRSN$ and $MRSQ$.

Then, $\angle ABC = \angle A'BC'$. (§ 39.)

Whence, diedral $PRSN$ = diedral $MRSQ$. (§ 431.)

EXERCISES.

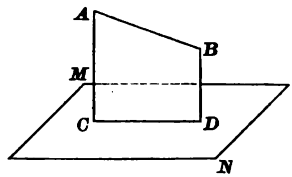
12. If two parallel planes are cut by a third plane, the alternate-interior diedrals are equal.

13. If a straight line is parallel to a plane, any plane perpendicular to the line is perpendicular to the plane.

14. If a plane be drawn through a diagonal of a parallelogram, the perpendiculars to it from the extremities of the other diagonal are equal.

Proposition XXIII. Theorem.

448. *Through a given straight line without a plane, a plane can be drawn perpendicular to the given plane, and but one.*



Let AB be the given line without the plane MN.

To prove that a plane can be drawn through AB perpendicular to MN, and but one.

Draw AC perpendicular to MN, and let AD be the plane determined by AB and AC.

Then, AD is perpendicular to MN. (§ 443.)

If more than one plane could be drawn through AB perpendicular to MN, their common intersection, AB, would be perpendicular to MN. (§ 445.)

Hence, but one plane can be drawn through AB perpendicular to MN, unless AB is perpendicular to MN.

Note. If the line AB is perpendicular to MN, an indefinitely great number of planes can be drawn through AB perpendicular to MN (§ 443).

449. Cor. *The projection of a straight line on a plane is a straight line.*

Let CD be the projection of the straight line AB on the plane MN.

To prove CD a straight line.

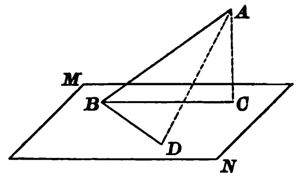
Let a plane be drawn through AB perpendicular to MN.

The perpendiculars to MN from all points of AB will lie in this plane. (§ 442.)

Therefore, CD is a straight line. (§ 396.)

PROPOSITION XXIV. THEOREM

450. *The angle between a straight line and its projection on a plane is the least angle which it makes with any line drawn in the plane through its foot.*



Let BC be the projection of the line AB on the plane MN.
Let BD be any other line drawn through B in MN.
To prove $\angle ABC < \angle ABD$.

Lay off $BD = BC$, and draw AC and AD.
Then in the triangles ABC and ABD, AB is common.
Also, $AC < AD$. (§ 402.)
Whence, $\angle ABC < \angle ABD$. (§ 90.)

451. SCH. ABC is called the *angle* between AB and MN.

EXERCISES.

15. If two parallels meet a plane, they make equal angles with it.

16. If a straight line intersects two parallel planes, it makes equal angles with them.

17. The angle between perpendiculars to the faces of a diedral from any point within the angle is the supplement of its plane angle.

18. If BC is the projection of the line AB upon the plane MN, and BD and BE be drawn in the plane making $\angle CBD = \angle CBE$, prove that $\angle ABD = \angle ABE$.

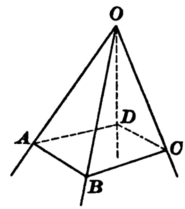
19. If each of two intersecting planes be cut by two parallel planes, not parallel to their intersection, their intersections with the parallel planes include equal angles.

20. The line AB is perpendicular to the plane MN at B. A line is drawn from B meeting the line CD of the plane MN at E. If AE is perpendicular to CD, prove that BE is perpendicular to CD.

POLYEDRALS.

Definitions.

452. If three or more planes meet in a common point, the figure formed is called a *polyedral angle*, or simply a *polyedral*.



The common point is called the *vertex* of the polyedral, and the intersections of the planes the *edges*.

The portions of the planes included between the edges are called the *faces* of the polyedral, and the angles formed by the edges are called the *face angles*.

Thus, in the polyedral *O–ABCD*, *O* is the vertex; *OA*, *OB*, etc., are the edges; the planes *AOB*, *BOC*, etc., are the faces; and the angles *AOB*, *BOC*, etc., are the face angles.

453. A polyedral must have at least three faces.

A polyedral of three faces is called a *triedral*.

454. To show more distinctly the relative positions of the edges of a polyedral, it is customary to represent them as intersected by a plane, as shown in the figure of § 452.

The plane *ABCD* is called the *base* of the polyedral.

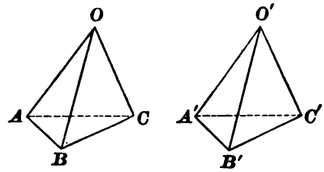
455. The polyedral is not regarded as limited by the base; thus, the face *AOB* is understood to mean, not the triangle *AOB*, but the indefinite plane included between the edges *OA* and *OB* produced indefinitely.

456. A polyedral is called *convex* when its base is a convex polygon (§ 120).

457. Two polyedrals are called *vertical* when the edges of one are the prolongations of the edges of the other.

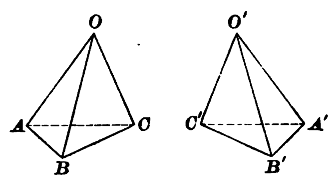
458. Two polyedrals are *equal* when they can be applied to each other so that their faces shall coincide.

459. Two polyedrals are equal when the face angles and diedrals of one are equal respectively to the homologous face angles and diedrals of the other, if the equal parts are arranged *in the same order*.



Thus, if the face angles AOB, BOC, and COA are equal respectively to the face angles $A'O'B'$, $B'O'C'$, and $C'O'A'$, and the diedrals OA, OB, and OC to the diedrals $O'A'$, $O'B'$, and $O'C'$, the triedrals O–ABC and O'–$A'B'C'$ are equal; for they can evidently be applied to each other so that their faces shall coincide.

460. Two polyedrals are said to be *symmetrical* when the face angles and diedrals of one are equal respectively to the homologous face angles and diedrals of the other, if the equal parts are arranged *in the reverse order*.



Thus, if the face angles AOB, BOC, and COA are equal respectively to the face angles $A'O'B'$, $B'O'C'$, and $C'O'A'$, and the diedrals OA, OB, and OC to the diedrals $O'A'$, $O'B'$, and $O'C'$, the triedrals O–ABC and O'–$A'B'C'$ are symmetrical.

461. It is evident that, in general, two symmetrical polyedrals cannot be placed so that their faces shall coincide.

EXERCISES.

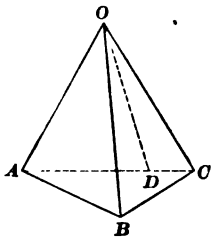
21. The three planes bisecting the diedrals of a triedral meet in a common straight line.

22. D is any point in the perpendicular AF from A to the side BC of the triangle ABC. If DE be drawn perpendicular to the plane of ABC, and GH be drawn through E parallel to BC, prove that AE is perpendicular to GH. (§ 398.)

Proposition XXV. Theorem.

462. *The sum of any two face angles of a triedral is greater than the third.*

Note. The theorem requires proof only in the case where the third angle is greater than either of the others.



In the triedral O–ABC, let the face angle AOC be greater than either AOB or BOC.

To prove $\angle AOB + \angle BOC > \angle AOC$.

In the face AOC, draw the line OD equal to OB, making $\angle AOD = \angle AOB$; and through B and D pass a plane cutting the faces of the triedral in AB, BC, and CA.

Then in the triangles AOB and AOD, OA is common.

And by construction, $OB = OD$,

and $\angle AOB = \angle AOD$.

Therefore, $\triangle AOB = \triangle AOD$. (§ 63.)

Whence, $AB = AD$. (§ 66.)

Now, $AB + BC > AD + DC$. (Ax. 6.)

Or, since $AB = AD$, $BC > DC$.

Then in the triangles BOC and COD, OC is common.

Also, $OB = OD$, and $BC > CD$.

Whence, $\angle BOC > \angle COD$. (§ 90.)

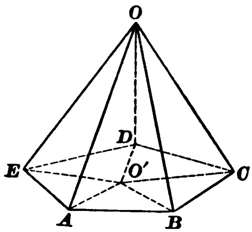
Adding $\angle AOB$ to the first member of this inequality, and its equal $\angle AOD$ to the second member, we have

$$\angle AOB + \angle BOC > \angle AOD + \angle COD.$$

Whence, $\angle AOB + \angle BOC > \angle AOC$.

PROPOSITION XXVI. THEOREM.

463. *The sum of the face angles of any convex polyedral is less than four right angles.*



Let $O\text{–}ABCDE$ be a convex polyedral.

To prove $\angle AOB + \angle BOC +$ etc. $<$ four right angles.

Let $ABCDE$ be the base of the polyedral.

Let O' be any point within the polygon $ABCDE$, and draw $O'A$, $O'B$, $O'C$, $O'D$, and $O'E$.

Then, $\angle OAE + \angle OAB > \angle O'AE + \angle O'AB$. (§ 462.)

In like manner,

$$\angle OBA + \angle OBC > \angle O'BA + \angle O'BC; \text{ etc.}$$

Adding these inequalities, we have the sum of the angles at the bases of the triangles whose common vertex is O greater than the sum of the angles at the bases of the triangles whose common vertex is O'.

But the sum of *all* the angles of the triangles whose common vertex is O is equal to the sum of *all* the angles of the triangles whose common vertex is O'. (§ 82.)

Hence, the sum of the angles at O is less than the sum of the angles at O'.

Therefore, the sum of the angles at O is less than four right angles. (§ 37.)

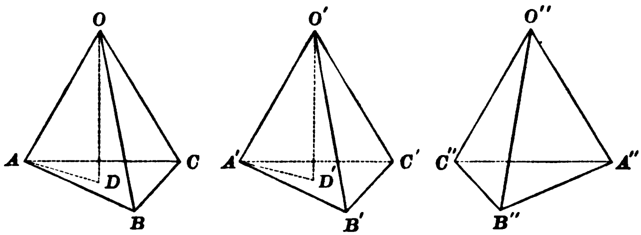
Ex. 23. Between two straight lines not in the same plane a common perpendicular can be drawn. (Ex. 11.)

PROPOSITION XXVII. THEOREM.

464. *If two triedrals have the face angles of one equal respectively to the face angles of the other,*

I. *They are equal if the equal parts occur in the same order.*

II. *They are symmetrical if the equal parts occur in the reverse order.*



I. In the triedrals $O\text{-}ABC$ and $O'\text{-}A'B'C'$, let

$$\angle AOB = \angle A'O'B',\ \angle BOC = \angle B'O'C',$$
$$\text{and } \angle COA = \angle C'O'A'.$$

To prove triedral $O\text{-}ABC$ = triedral $O'\text{-}A'B'C'$.

Lay off the six equal distances OA, OB, OC, $O'A'$, $O'B'$, and $O'C'$; and draw AB, BC, CA, $A'B'$, $B'C'$, and $C'A'$.

Then, $\triangle OAB = \triangle O'A'B'$. (§ 63.)

Whence, $AB = A'B'$. (§ 66.)

Similarly, $BC = B'C'$, and $CA = C'A'$.

Therefore, $\triangle ABC = \triangle A'B'C'$. (§ 69.)

Draw OD and $O'D'$ perpendicular to ABC and $A'B'C'$, respectively; also, draw AD and $A'D'$.

The equal oblique lines OA, OB, and OC meet the plane ABC at equal distances from D. (§ 408.)

Hence, D is the centre of the circumscribed circle of the triangle ABC; and similarly, D' is the centre of the circumscribed circle of $A'B'C'$.

Now apply $O'\text{-}A'B'C'$ to $O\text{-}ABC$, so that the points A', B', and C' shall fall at A, B, and C, and the point D' at D.

Then the perpendicular $O'D'$ will fall upon OD. (§ 399.)

But the right triangles OAD and $O'A'D'$ are equal. (§ 88.)

Whence, $O'D' = OD$, and the point O' will fall at O.

Therefore, the triedrals $O\text{–}ABC$ and $O'\text{–}A'B'C'$ coincide throughout, and are equal.

II. In the triedrals $O\text{–}ABC$ and $O''\text{–}A''B''C''$, let the angles AOB, BOC, and COA be equal respectively to $A''O''B''$, $B''O''C''$, and $C''O''A''$.

To prove $O\text{–}ABC$ symmetrical to $O''\text{–}A''B''C''$.

Construct $O'\text{–}A'B'C'$ symmetrical to $O''\text{–}A''B''C''$, having the angles $A'O'B'$, $B'O'C'$, and $C'O'A'$ equal respectively to $A''O''B''$, $B''O''C''$, and $C''O''A''$.

Then the triedrals $O\text{–}ABC$ and $O'\text{–}A'B'C'$ have the angles AOB, BOC, and COA equal respectively to $A'O'B'$, $B'O'C'$, and $C'O'A'$.

Hence, triedral $O\text{–}ABC$ = triedral $O'\text{–}A'B'C'$. (§ 464, I.)

Therefore, $O\text{–}ABC$ is symmetrical to $O''\text{–}A''B''C''$.

465. Cor. *If two triedrals have the face angles of one equal respectively to the face angles of the other, their homologous diedrals are equal.*

EXERCISES.

24. Two triedrals are equal when two face angles and the included diedral of one are equal respectively to two face angles and the included diedral of the other, and similarly arranged.

25. Two triedrals are equal when a face angle and the adjacent diedrals of one are equal respectively to a face angle and the adjacent diedrals of the other, and similarly arranged.

26. A is any point in the face EG of the diedral $DEFG$. If AC be drawn perpendicular to the edge EF, and AB perpendicular to the face DF, prove that the plane determined by AC and BC is perpendicular to EF.

27. From any point E within the diedral $CABD$, EF and EG are drawn perpendicular to the faces ABC and ABD, and GH perpendicular to the face ABC at H. Prove FH perpendicular to AB.

BOOK VII.

POLYEDRONS.

DEFINITIONS.

466. A *polyedron* is a solid bounded by planes.

The bounding planes are called the *faces* of the polyedron; their intersections are called the *edges*, and the intersections of the edges the *vertices*.

A *diagonal* is a straight line joining any two vertices not in the same face.

467. The least number of planes which can form a polyedral is three (§ 453); hence, the least number of planes which can bound a polyedron is four.

A polyedron of four faces is called a *tetraedron*; of six faces, a *hexaedron*; of eight faces, an *octaedron*; of twelve faces, a *dodecaedron*; of twenty faces, an *icosaedron*.

468. A polyedron is called *convex* when the section made by any plane is a convex polygon (§ 120).

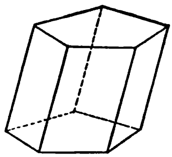
All polyedrons considered hereafter will be understood to be convex.

469. The *volume* of a solid is its ratio to another solid, called the *unit of volume*, adopted arbitrarily as the unit of measure (§ 179).

470. Two solids are said to be *equivalent* when their volumes are equal.

PRISMS AND PARALLELOPIPEDS.

471. A *prism* is a polyedron, two of whose faces are equal polygons lying in parallel planes, having their homologous sides parallel, the other faces being parallelograms.



The equal and parallel faces are called the *bases* of the prism, and the remaining faces the *lateral faces*; the intersections of the lateral faces are called the *lateral edges*, and the sum of the areas of the lateral faces the *lateral area*.

The *altitude* is the perpendicular distance between the planes of the bases.

472. The following is given for convenience of reference:

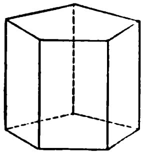
The bases of a prism are equal.

473. It follows from the definition of § 471 that

The lateral edges of a prism are equal and parallel.

474. A prism is called *triangular*, *quadrangular*, etc., according as its base is a triangle, quadrilateral, etc.

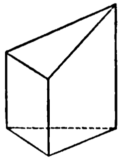
475. A *right prism* is a prism whose lateral edges are perpendicular to its bases.



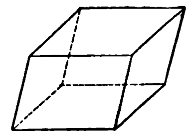
An *oblique prism* is a prism whose lateral edges are not perpendicular to its bases.

476. A *regular prism* is a right prism whose base is a regular polygon.

477. A *truncated prism* is that portion of a prism included between the base, and a plane, not parallel to the base, cutting all the lateral edges.

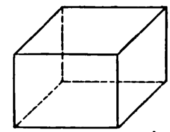


478. A *right section* of a prism is the section made by a plane perpendicular to the lateral edges.



479. A *parallelopiped* is a prism whose bases are parallelograms; that is, all the faces are parallelograms.

480. A *right parallelopiped* is a parallelopiped whose lateral edges are perpendicular to its bases.



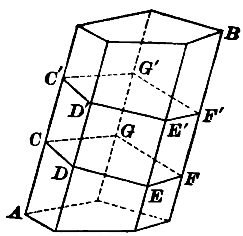
481. A *rectangular parallelopiped* is a right parallelopiped whose bases are rectangles; that is, all the faces are rectangles.

The *dimensions* are the three edges which meet at any vertex.

482. A *cube* is a rectangular parallelopiped whose six faces are all squares.

PROPOSITION I. THEOREM.

483. *The sections of a prism made by two parallel planes which cut all the lateral edges, are equal polygons.*



Let the parallel planes CF and $C'F'$ cut all the lateral edges of the prism AB.

To prove that the sections $CDEFG$ and $C'D'E'F'G'$ are equal.

We have CD parallel to $C'D'$, DE to $D'E'$, etc. (§ 417.)

Whence, $CD = C'D'$, $DE = D'E'$, etc. (§ 105.)

Then the polygons $CDEFG$ and $C'D'E'F'G'$ are mutually equilateral.

Again, $\angle CDE = \angle C'D'E'$,

$\angle DEF = \angle D'E'F'$, etc. (§ 424.)

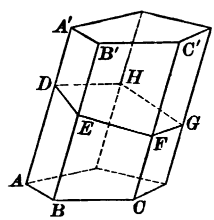
Then the polygons $CDEFG$ and $C'D'E'F'G'$ are mutually equiangular.

Therefore, $CDEFG$ and $C'D'E'F'G'$ are equal. (§ 124.)

484. Cor. *The section of a prism made by a plane parallel to the base is equal to the base.*

Proposition II. Theorem.

485. *The lateral area of a prism is equal to the perimeter of a right section multiplied by a lateral edge.*



Let $DEFGH$ be a right section of the prism AC'.

To prove

$$\text{lat. area } AC' = (DE + EF + \text{etc.}) \times AA'.$$

We have DE perpendicular to AA'. (§ 398.)

Whence, area $AA'B'B = DE \times AA'$. (§ 310.)

Similarly, area $BB'C'C = EF \times BB'$

$= EF \times AA'$; etc. (§ 473.)

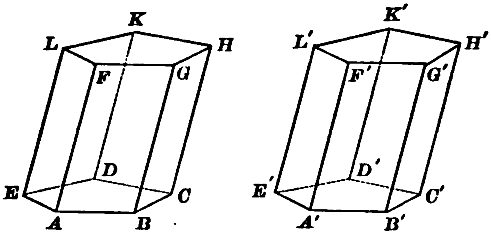
Adding these equations, we have

$$\begin{aligned}\text{lat. area } AC' &= DE \times AA' + EF \times AA' + \text{etc.}\\ &= (DE + EF + \text{etc.}) \times AA'.\end{aligned}$$

486. Cor. *The lateral area of a right prism is equal to the perimeter of the base multiplied by the altitude.*

Proposition III. Theorem.

487. *Two prisms are equal when the faces including a triedral of one are equal respectively to the faces including a triedral of the other, and similarly placed.*



In the prisms AH and $A'H'$, let the faces $ABCDE$, AG, and AL be equal respectively to the faces $A'B'C'D'E'$, $A'G'$, and $A'L'$; the equal parts being similarly placed.

To prove the prisms equal.

The angles EAB, EAF, and FAB are equal respectively to the angles $E'A'B'$, $E'A'F'$, and $F'A'B'$.

Then, triedral $A\text{–}BEF$ = triedral $A'\text{–}B'E'F'$. (§ 464, I.)

Then the prism $A'H'$ may be applied to AH so that the vertices A', B', C', D', E', G', F', and L' shall fall at A, B, C, D, E, G, F, and L, respectively.

Now since the lateral edges of the prisms are parallel, the edge $C'H'$ will fall upon CH, and $D'K'$ upon DK.

And since the points G', F', and L' fall at G, F, and L, the planes of the upper bases will coincide. (§ 395, II.)

Therefore, the points H' and K' fall at H and K.

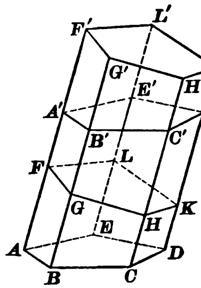
Hence, the prisms coincide throughout, and are equal.

488. Sch. The above demonstration applies without change to the case of two *truncated prisms.*

489. Cor. *Two right prisms are equal when they have equal bases and equal altitudes;* for by inverting one of the prisms if necessary, the equal faces will be similarly placed.

Proposition IV. Theorem.

490. *An oblique prism is equivalent to a right prism, having for its base a right section of the oblique prism, and for its altitude a lateral edge of the oblique prism.*



Let $FGHKL$ be a right section of the oblique prism AD'.

Produce AA' to F', making $FF' = AA'$.

At F' pass the plane $F'K'$ parallel to $FGHKL$, meeting the edges BB', CC', etc., produced at G', H', etc.

To prove AD' equivalent to the right prism FK'.

In the truncated prisms AK and $A'K'$, the faces $FGHKL$ and $F'G'H'K'L'$ are equal. (§ 472.)

Therefore, $A'K'$ may be applied to AK so that the vertices F', G', etc., shall fall at F, G, etc., respectively.

Then the edges $A'F'$, $B'G'$, etc., will coincide in direction with AF, BG, etc. (§ 399.)

But, since $FF' = AA'$, we have $AF = A'F'$.

In like manner, $BG = B'G'$, $CH = C'H'$, etc.

Hence, the vertices A', B', etc., will fall at A, B, etc.

Then, $A'K'$ and AK coincide throughout, and are equal.

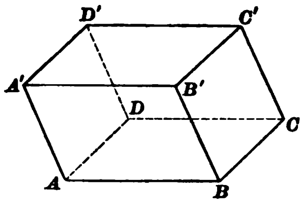
Now taking from the entire solid AK' the truncated prism $A'K'$, there remains the prism AD'.

And taking its equal AK, there remains the prism FK'.

Hence, AD' and FK' are equivalent.

Proposition V. Theorem.

491. ***The opposite faces of a parallelopiped are equal and parallel.***



Let AC and $A'C'$ be the bases of the parallelopiped AC'.

To prove the faces AB' and DC' equal and parallel.

AB is equal and parallel to DC, and AA' to DD'. (§ 104.)

Hence, $\angle A'AB = \angle D'DC$,

and the faces AB' and DC' are parallel. (§ 424.)

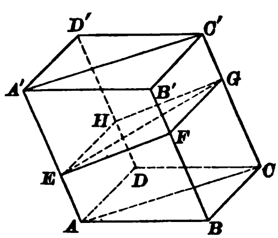
Therefore, the faces AB' and DC' are equal. (§ 112.)

In like manner, we may prove AD' and BC' equal and parallel.

492. Cor. *Either face of a parallelopiped may be taken as the base.*

Proposition VI. Theorem.

493. ***The plane passed through two diagonally opposite edges of a parallelopiped divides it into two equivalent triangular prisms.***



Let AC' be a parallelopiped.

Through the edges AA' and CC' pass a plane dividing AC' into two triangular prisms, $ABC\text{–}A'$ and $ACD\text{–}A'$.

To prove $ABC\text{–}A' \Leftrightarrow ACD\text{–}A'$.

Let $EFGH$ be a right section of the parallelopiped, cutting the plane $AA'C'C$ in EG.

Now the planes AB' and DC' are parallel. (§ 491.)

Whence, EF is parallel to GH. (§ 417.)

In like manner, EH is parallel to FG.

Therefore, $EFGH$ is a parallelogram.

Whence, $\triangle EFG = \triangle EGH$. (§ 106.)

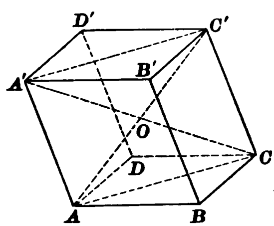
Now, $ABC\text{–}A'$ is equivalent to a right prism whose base is EFG, and altitude AA'; and $ACD\text{–}A'$ is equivalent to a right prism whose base is EGH, and altitude AA'. (§ 490.)

But these two right prisms are equal. (§ 489.)

Therefore, $ABC\text{–}A' \Leftrightarrow ACD\text{–}A'$.

Proposition VII. Theorem.

494. *The diagonals of a parallelopiped bisect each other.*



Let AC' and $A'C$ be diagonals of the parallelopiped AC'.

To prove that AC' and $A'C$ bisect each other.

Draw AC and $A'C'$.

Then AA' is equal and parallel to CC'. (§ 473.)

Whence, the figure $AA'C'C$ is a parallelogram. (§ 109.)

Therefore, AC' and $A'C$ bisect each other at O. (§ 110.)

In like manner, we may prove that any two of the four diagonals AC', $A'C$, BD', and $B'D$ bisect each other at O.

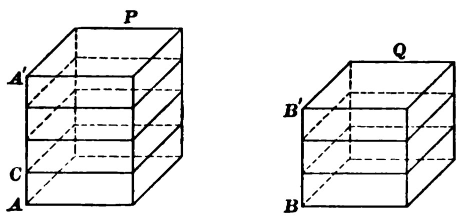
Note. The point O is called the *centre* of the parallelopiped.

Proposition VIII. Theorem.

495. ***Two rectangular parallelopipeds having equal bases are to each other as their altitudes.***

Note. **The phrase "rectangular parallelopiped" in the above statement signifies the *volume* of the rectangular parallelopiped.**

Case I. ***When the altitudes are commensurable.***



Let P and Q be two rectangular parallelopipeds, having equal bases, and commensurable altitudes AA' and BB'.

To prove $$\frac{P}{Q} = \frac{AA'}{BB'}.$$

Let AC be a common measure of AA' and BB', and let it be contained 4 times in AA', and 3 times in BB'.

Then, $$\frac{AA'}{BB'} = \frac{4}{3}. \qquad (1)$$

At the several points of division of AA' and BB' pass planes perpendicular to these lines.

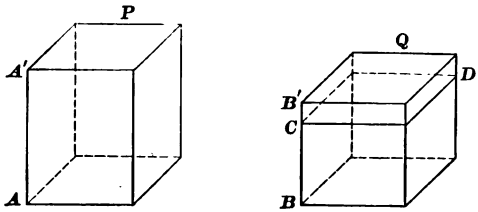
Then the parallelopiped P will be divided into 4 parts, and the parallelopiped Q into 3 parts, all of which parts will be equal. (§ 489.)

Therefore, $$\frac{P}{Q} = \frac{4}{3}. \qquad (2)$$

From (1) and (2), we have

$$\frac{P}{Q} = \frac{AA'}{BB'}.$$

CASE II. *When the altitudes are incommensurable.*



Let P and Q be two rectangular parallelopipeds, having equal bases, and incommensurable altitudes AA' and BB'.

To prove $$\frac{P}{Q} = \frac{AA'}{BB'}.$$

Let AA' be divided into any number of equal parts, and let one of these parts be applied to BB' as a measure.

Since AA' and BB' are incommensurable, a certain number of the parts will extend from B to C, leaving a remainder CB' less than one of the parts.

Pass the plane CD perpendicular to BB', and let Q' denote the rectangular parallelopiped BD.

Then since AA' and BC are commensurable,

$$\frac{P}{Q'} = \frac{AA'}{BC}. \qquad \text{(§ 495, Case I.)}$$

Now let the number of subdivisions of AA' be indefinitely increased.

Then the length of each part will be indefinitely diminished, and the remainder CB' will approach the limit 0.

Then, $\frac{P}{Q'}$ will approach the limit $\frac{P}{Q}$,

and $\frac{AA'}{BC}$ will approach the limit $\frac{AA'}{BB'}$.

By the Theorem of Limits, these limits are equal. (§ 188.)

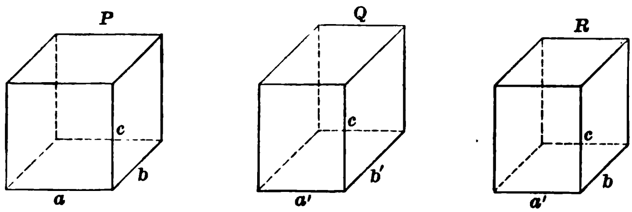
Whence, $$\frac{P}{Q} = \frac{AA'}{BB'}.$$

496. Sch. The theorem may also be expressed:

Two rectangular parallelopipeds having two dimensions in common, are to each other as their third dimensions.

Proposition IX. Theorem.

497. *Two rectangular parallelopipeds having equal altitudes are to each other as their bases.*



Let P and Q be two rectangular parallelopipeds, having the dimensions a, b, c, and a', b', c, respectively.

To prove $$\frac{P}{Q}=\frac{a\times b}{a'\times b'}.$$

Let R be a rectangular parallelopiped having the dimensions a', b, and c.

Then P and R have the dimensions b and c in common.

Whence, $$\frac{P}{R}=\frac{a}{a'}.\qquad\text{(§ 496.)}$$

And R and Q have the dimensions a' and c in common.

Whence, $$\frac{R}{Q}=\frac{b}{b'}.$$

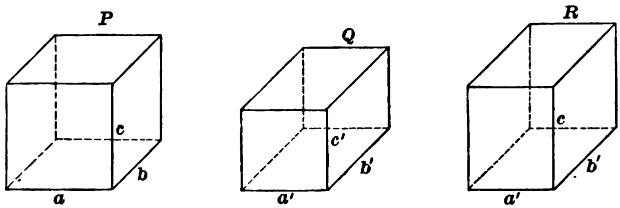
Multiplying these equations, we have
$$\frac{P}{Q}=\frac{a\times b}{a'\times b'}.$$

498. Sch. The theorem may also be expressed:

Two rectangular parallelopipeds having one dimension in common, are to each other as the products of their other two dimensions.

PROPOSITION X. THEOREM.

499. *Any two rectangular parallelopipeds are to each other as the products of their three dimensions.*



Let P and Q be two rectangular parallelopipeds, having the dimensions a, b, c, and a', b', c', respectively.

To prove $$\frac{P}{Q} = \frac{a \times b \times c}{a' \times b' \times c'}.$$

Let R be a rectangular parallelopiped having the dimensions a', b', and c.

Then P and R have the dimension c in common.

Whence, $$\frac{P}{R} = \frac{a \times b}{a' \times b'}. \qquad (\S\ 498.)$$

And R and Q have the dimensions a' and b' in common.

Whence, $$\frac{R}{Q} = \frac{c}{c'}. \qquad (\S\ 496.)$$

Multiplying these equations, we have

$$\frac{P}{Q} = \frac{a \times b \times c}{a' \times b' \times c'}.$$

EXERCISES.

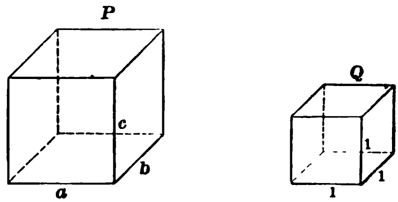
1. Two rectangular parallelopipeds have the dimensions 6, 8, and 14, and 7, 8, and 9, respectively. What is the ratio of their volumes?

2. Find the ratio of the volumes of two rectangular parallelopipeds, whose dimensions are 8, 12, and 21, and 14, 15, and 24, respectively.

3. The diagonals of a rectangular parallelopiped are equal.

Proposition XI. Theorem.

500. *If the unit of volume is the cube whose edge is the linear unit, the volume of a rectangular parallelopiped is equal to the product of its three dimensions.*



Let a, b, and c be the dimensions of the rectangular parallelopiped P; and let Q be the unit of volume, i.e., a cube whose edge is the linear unit.

To prove vol. $P = a \times b \times c$.

We have $\frac{P}{Q} = \frac{a \times b \times c}{1 \times 1 \times 1} = a \times b \times c.$ (§ 499.)

But since Q is the unit of volume,

$$\frac{P}{Q} = \text{vol. } P. \qquad (\S\ 469.)$$

Whence, vol. $P = a \times b \times c$.

501. Cor. I. *The volume of a cube is equal to the cube of its edge.*

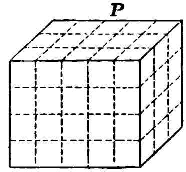
502. Cor. II. If c be taken as the altitude of the parallelopiped P, $a \times b$ is the area of its base. (§ 305.)

Hence, *the volume of a rectangular parallelopiped is equal to the product of its base and altitude.*

503. Sch. I. In all succeeding theorems relating to volumes, it is understood that the *unit of volume* is the cube whose edge is the linear unit, and the *unit of surface* the square whose side is the linear unit. (Compare § 307.)

504. SCH. II. If the dimensions of the rectangular parallelopiped are *multiples* of the linear unit, the truth of Prop. XI. may be seen by dividing the solid into cubes, each equal to the unit of volume.

P



Thus, if the dimensions of the rectangular parallelopiped P are 5 units, 4 units, and 3 units, respectively, the solid can evidently be divided into 60 cubes.

In this case, 60, the number which expresses the volume of the rectangular parallelopiped, is the product of 5, 4, and 3, the numbers which express the lengths of its edges.

EXERCISES.

4. Find the altitude of a rectangular parallelopiped, the dimensions of whose base are 21 and 30, equivalent to a rectangular parallelopiped whose dimensions are 27, 28, and 35.

5. Find the edge of a cube equivalent to a rectangular parallelopiped whose dimensions are 9 in., 1 ft. 9 in., and 4 ft. 1 in.

6. Find the volume, and the area of the entire surface, of a cube whose edge is $3\frac{1}{4}$ in.

7. Find the area of the entire surface of a rectangular parallelopiped, the dimensions of whose base are 11 and 13, and volume 858.

8. Find the volume of a rectangular parallelopiped, the dimensions of whose base are 14 and 9, and the area of whose entire surface is 620.

9. Find the dimensions of the base of a rectangular parallelopiped, the area of whose entire surface is 320, volume 336, and altitude 4.

10. How many bricks, each 8 in. long, $2\frac{3}{4}$ in. wide, and 2 in. thick, will be required to build a wall 18 ft. long, 3 ft. high, and 11 in. thick?

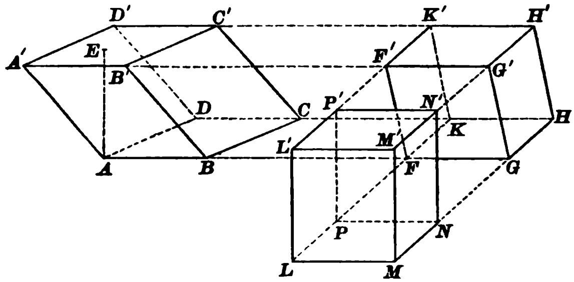
11. The section of a prism made by a plane parallel to a lateral edge is a parallelogram.

12. The square of a diagonal of a rectangular parallelopiped is equal to the sum of the squares of its dimensions.

13. Find the length of the diagonal of a rectangular parallelopiped whose dimensions are 8, 9, and 12.

Proposition XII. Theorem.

505. *The volume of any parallelopiped is equal to the product of its base and altitude.*



Let AE be the altitude of the parallelopiped AC'.

To prove vol. $AC' = ABCD \times AE$.

Produce the edges AB, $A'B'$, $D'C'$, and DC.

On AB produced, take $FG = AB$; and pass the planes FK' and GH' perpendicular to FG, forming the right parallelopiped FH'.

Then, FH' is equivalent to AC'. (§ 490.)

Produce the edges HG, $H'G'$, $K'F'$, and KF.

On HG produced, take $NM = HG$; and pass the planes NP' and ML' perpendicular to NM, forming the right parallelopiped LN'.

Then, LN' is equivalent to FH'. (§ 490.)

Whence, LN' is equivalent to AC'.

Now since FG is perpendicular to the plane GH', the planes LH and MH' are perpendicular. (§ 443.)

But LMM' is the plane angle of the diedral $LMHH'$. (§ 433.)

Whence, LMM' is a right angle. (§ 438.)

Therefore, LM' is a rectangle, and LN' is a rectangular parallelopiped.

Whence, vol. $LN' = LMNP \times MM'$. (§ 502.)

That is, vol. $AC' = LMNP \times MM'$. (1)

But the rectangle $LMNP$ is equal to the rectangle $FGHK$; for they have equal bases MN and GH, and the same altitude. (§ 113.)

And the rectangle $FGHK$ is equivalent to the parallelogram $ABCD$; for they have equal bases FG and AB, and the same altitude. (§ 311.)

Therefore, $LMNP$ is equivalent to $ABCD$.

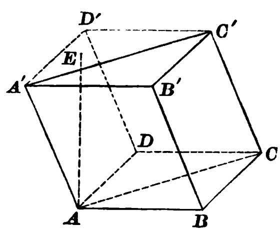
Again, $MM' = AE$. (§ 422.)

Substituting these values in (1), we have

$$\text{vol. } AC' = ABCD \times AE.$$

Proposition XIII. Theorem.

506. *The volume of a triangular prism is equal to the product of its base and altitude.*



Let AE be the altitude of the triangular prism $ABC\text{–}C'$.

To prove vol. $ABC\text{–}C' = ABC \times AE$.

Construct the parallelopiped $ABCD\text{–}D'$, having its edges parallel to AB, BC, and BB', respectively.

Then, vol. $ABC\text{–}C' = \frac{1}{2}$ vol. $ABCD\text{–}D'$ (§ 493.)

$= \frac{1}{2} ABCD \times AE$ (§ 505.)

$= ABC \times AE$. (§ 106.)

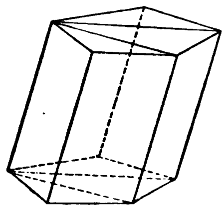
EXERCISES.

14. Find the lateral area and volume of a regular triangular prism, each side of whose base is 5, and whose altitude is 8.

15. The diagonal of a cube is equal to its edge multiplied by $\sqrt{3}$.

PROPOSITION XIV. THEOREM.

507. *The volume of any prism is equal to the product of its base and altitude.*



Any prism may be divided into triangular prisms by passing planes through one of the lateral edges and the corresponding diagonals of the base.

The volume of each triangular prism is equal to the product of its base and altitude (§ 506).

Hence, the sum of the volumes of the triangular prisms is equal to the sum of their bases multiplied by their common altitude.

Therefore, the volume of the given prism is equal to the product of its base and altitude.

508. COR. I. *Two prisms having equivalent bases and equal altitudes are equivalent.*

509. COR. II. 1. *Two prisms having equal altitudes are to each other as their bases.*

2. *Two prisms having equivalent bases are to each other as their altitudes.*

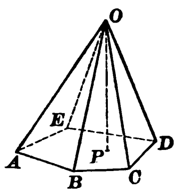
3. *Any two prisms are to each other as the products of their bases by their altitudes.*

Ex. 16. Find the lateral area and volume of a regular hexagonal prism, each side of whose base is 3, and whose altitude is 9.

PYRAMIDS.

DEFINITIONS.

510. A *pyramid* is a polyedron bounded by a polygon, and a series of triangles having a common vertex; as *O–ABCDE.*



The polygon is called the *base* of the pyramid, and the common vertex of the triangular faces is called the *vertex*.

The triangular faces are called the *lateral faces,* and their intersections the *lateral edges.*

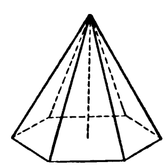
The sum of the areas of the lateral faces is called the *lateral area.*

511. The *altitude* of a pyramid is the perpendicular distance from the vertex to the plane of the base.

512. A pyramid is called *triangular, quadrangular,* etc., according as its base is a triangle, quadrilateral, etc.

NOTE. A triangular pyramid is a tetraedron (§ 467).

513. A *regular pyramid* is a pyramid whose base is a regular polygon, and whose vertex lies in the perpendicular erected at the centre of the base.



514. The lateral edges of a regular pyramid are equal. (§ 407, I.)

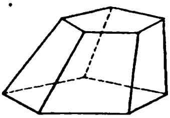
Whence, the lateral faces are equal isosceles triangles. (§ 69.)

515. The *slant height* of a regular pyramid is the perpendicular distance from the vertex of the pyramid to any side of the base.

Or, it is the straight line drawn from the vertex of the pyramid to the middle point of any side of the base. (§ 92.)

516. A *truncated pyramid* is that portion of a pyramid included between the base, and a plane cutting all the lateral edges.

517. A *frustum of a pyramid* is a truncated pyramid whose bases are parallel.



The *altitude* of the frustum is the perpendicular distance between the planes of its bases.

The lateral faces of a frustum of a pyramid are trapezoids. (§ 417.)

518. The lateral faces of a frustum of a regular pyramid are equal.

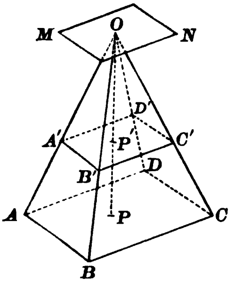
The *slant height* of a frustum of a regular pyramid is the altitude of any lateral face.

Proposition XV. Theorem.

519. *If a pyramid be cut by a plane parallel to its base,*

I. *The lateral edges and the altitude are divided proportionally.*

II. *The section is similar to the base.*



Let $A'C'$ be a plane parallel to the base of the pyramid $O\text{–}ABCD$, cutting the faces OAB, OBC, OCD, and ODA in the lines $A'B'$, $B'C'$, $C'D'$, and $D'A'$, and the altitude OP at P'.

I. To prove $\frac{OA'}{OA} = \frac{OB'}{OB} = \frac{OC'}{OC}$ etc. $= \frac{OP'}{OP}$.

Through O pass the plane MN parallel to $ABCD$.

Then, $$\frac{OA'}{OA} = \frac{OB'}{OB} = \frac{OC'}{OC} \text{ etc.} = \frac{OP'}{OP}. \quad (\S\ 425.)$$

II. To prove the section $A'B'C'D'$ similar to $ABCD$.

We have $A'B'$ parallel to AB, $B'C'$ to BC, etc. (§ 417.)

Then, $$\angle A'B'C' = \angle ABC,$$
$$\angle B'C'D' = \angle BCD, \text{ etc.} \quad (\S\ 424.)$$

That is, the polygons $A'B'C'D'$ and $ABCD$ are mutually equiangular.

Again, the triangles $OA'B'$ and OAB are similar. (§ 258.)

Whence, $$\frac{OA'}{OA} = \frac{A'B'}{AB}. \quad (1)$$

In like manner, $$\frac{OB'}{OB} = \frac{B'C'}{BC}, \text{ etc.}$$

But, $$\frac{OA'}{OA} = \frac{OB'}{OB}, \text{ etc.} \quad (\S\ 519, \text{I.})$$

Whence, $$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'D'}{CD}, \text{ etc.}$$

That is, the polygons $A'B'C'D'$ and $ABCD$ have their homologous sides proportional.

Therefore, $A'B'C'D'$ and $ABCD$ are similar. (§ 252.)

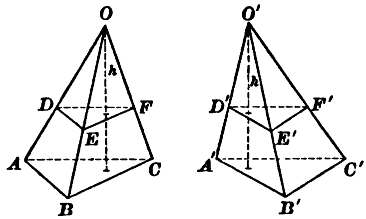
520. Cor. I. We have
$$\frac{\text{area } A'B'C'D'}{\text{area } ABCD} = \frac{\overline{A'B'}^2}{\overline{AB}^2}.$$

But, from equation (1) of § 519,
$$\frac{A'B'}{AB} = \frac{OA'}{OA} = \frac{OP'}{OP}. \quad (\S\ 519, \text{I.})$$

Whence, $$\frac{\text{area } A'B'C'D'}{\text{area } ABCD} = \frac{\overline{OP'}^2}{\overline{OP}^2}.$$

That is, *the areas of two parallel sections of a pyramid are to each other as the squares of their distances from the vertex.*

521. COR. II. ***If two pyramids have equal altitudes and equivalent bases, sections parallel to their bases equally distant from their vertices are equivalent.***



Let the bases of the pyramids O–ABC and O'–$A'B'C'$ be equivalent, and let the altitude of each pyramid be H.

Let DEF and $D'E'F'$ be sections parallel to the bases, at the distance h from the vertices.

To prove DEF equivalent to $D'E'F'$.

Now, $\dfrac{\text{area } DEF}{\text{area } ABC} = \dfrac{h^2}{H^2}$, and $\dfrac{\text{area } D'E'F'}{\text{area } A'B'C'} = \dfrac{h^2}{H^2}$. (§ 520.)

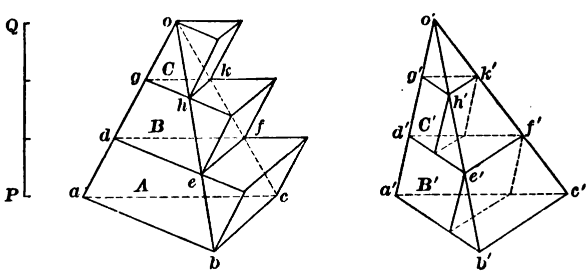
Whence, $$\frac{\text{area } DEF}{\text{area } ABC} = \frac{\text{area } D'E'F'}{\text{area } A'B'C'}.$$

But by hypothesis, area ABC = area $A'B'C'$.

Therefore, area DEF = area $D'E'F'$.

PROPOSITION XVI. THEOREM.

522. ***Two triangular pyramids having equal altitudes and equivalent bases are equivalent.***



Let o–abc and o'–$a'b'c'$ be two triangular pyramids having equal altitudes and equivalent bases.

To prove $\quad$ vol. $o\text{–}abc =$ vol. $o'\text{–}a'b'c'$.

Place the pyramids with their bases in the same plane, and let PQ be their common altitude.

Divide PQ into any number of equal parts; and through the points of division pass planes parallel to the plane of the bases, cutting $o\text{–}abc$ in the sections def and ghk, and $o'\text{–}a'b'c'$ in the sections $d'e'f'$ and $g'h'k'$.

Then def is equivalent to $d'e'f'$, and ghk to $g'h'k'$. (§ 521.)

With abc, def, and ghk as *lower* bases, construct the prisms A, B, and C, having their lateral edges equal and parallel to ad; and with $d'e'f'$ and $g'h'k'$ as *upper* bases, construct the prisms B' and C', having their lateral edges equal and parallel to $a'd'$.

Then, the prism B is equivalent to B'. (§ 508.)

In like manner, C is equivalent to C'.

Hence, the sum of the prisms circumscribed about $o\text{–}abc$ exceeds the sum of the prisms inscribed in $o'\text{–}a'b'c'$ by the prism A.

But $o\text{–}abc$ is evidently less than the sum of the prisms A, B, and C; and it is greater than the sum of the inscribed prisms, equivalent to B' and C', which can be constructed with def and ghk as *upper* bases.

Again, $o'\text{–}a'b'c'$ is greater than the sum of the prisms B' and C'; and it is less than the sum of the circumscribed prisms, equivalent to A, B, and C, which can be constructed with $a'b'c'$, $d'e'f'$, and $g'h'k'$ as *lower* bases.

Hence, the difference of the volumes of the pyramids must be less than the difference of the volumes of the two systems of prisms, and must therefore be less than the volume of the prism A.

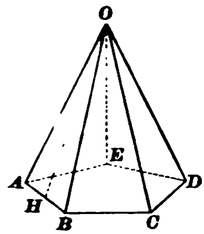
Now by sufficiently increasing the number of subdivisions of PQ, the volume of the prism A may be made less than any assigned volume, however small.

Therefore, the volumes of the pyramids cannot differ by any volume, however small.

Whence, $\quad$ vol. $o\text{–}abc =$ vol. $o'\text{–}a'b'c'$.

PROPOSITION XVII. THEOREM.

523. *The lateral area of a regular pyramid is equal to the perimeter of its base multiplied by one-half its slant height.*



Let OH be the slant height of the regular pyramid O–$ABCDE$.

To prove

lat. area $O\text{–}ABCDE = (AB + BC + \text{etc.}) \times \frac{1}{2}\,OH$.

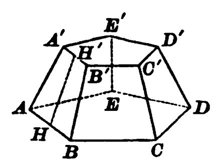
Now, area $OAB = AB \times \frac{1}{2}\,OH$. (§ 313.)

Also, area $OBC = BC \times \frac{1}{2}\,OH$; etc. (§ 515.)

Adding these equations, we have

lat. area $O\text{–}ABCDE = (AB + BC + \text{etc.}) \times \frac{1}{2}\,OH$.

524. COR. *The lateral area of a frustum of a regular pyramid is equal to one-half the sum of the perimeters of its bases, multiplied by its slant height.*



Let HH' be the slant height of the frustum of a regular pyramid AD'.

To prove

lat. area $AD' = \frac{1}{2}(AB + A'B' + BC + B'C' + \text{etc.}) \times HH'$.

Now, area $AA'B'B = \frac{1}{2}(AB + A'B') \times HH'$. (§ 317.)

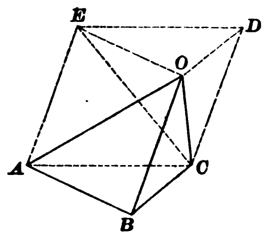
Also, area $BB'C'C = \frac{1}{2}(BC + B'C') \times HH'$; etc.

Adding these equations, we have

lat. area $AD' = \frac{1}{2}(AB + A'B' + BC + B'C' + \text{etc.}) \times HH'$.

Proposition XVIII. Theorem.

525. *A triangular pyramid is equivalent to one-third of a triangular prism having the same base and altitude.*



Let $O\text{–}ABC$ be a triangular pyramid.

Upon the base ABC, construct the prism $ABC\text{–}ODE$, having its lateral edges equal and parallel to OB.

To prove vol. $O\text{–}ABC = \frac{1}{3}$ vol. $ABC\text{–}ODE$.

The prism $ABC\text{–}ODE$ is composed of the triangular pyramid $O\text{–}ABC$, and the quadrangular pyramid $O\text{–}ACDE$.

Divide $O\text{–}ACDE$ into two triangular pyramids, $O\text{–}ACE$ and $O\text{–}CDE$, by passing a plane through O, C, and E.

Now, $O\text{–}ACE$ and $O\text{–}CDE$ have the same altitude.

And since CE is a diagonal of the parallelogram $ACDE$ they have equal bases, ACE and CDE. (§ 106.)

Hence, vol. $O\text{–}ACE =$ vol. $O\text{–}CDE$. (§ 522.)

Again, the pyramid $O\text{–}CDE$ may be regarded as having its vertex at C, and the triangle ODE for its base.

Then, the pyramids $O\text{–}ABC$ and $C\text{–}ODE$ have the same altitude. (§ 422.)

They have also equal bases, ABC and ODE. (§ 472.)

Hence, vol. $O\text{–}ABC =$ vol. $C\text{–}ODE$. (§ 522.)

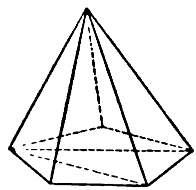
Then, vol. $O\text{–}ABC =$ vol. $O\text{–}ACE =$ vol. $O\text{–}CDE$.

Whence, vol. $O\text{–}ABC = \frac{1}{3}$ vol. $ABC\text{–}ODE$.

526. Cor. *The volume of a triangular pyramid is equal to one-third the product of its base and altitude.* (§ 506.)

Proposition XIX. Theorem.

527. *The volume of any pyramid is equal to one-third the product of its base and altitude.*



Any pyramid may be divided into triangular pyramids by passing planes through one of the lateral edges and the corresponding diagonals of the base.

The volume of each triangular pyramid is equal to one-third the product of its base and altitude (§ 526).

Hence, the sum of the volumes of the triangular pyramids is equal to the sum of their bases multiplied by one-third their common altitude.

Therefore, the volume of the given pyramid is equal to one-third the product of its base and altitude.

528. Cor. I. *Two pyramids having equivalent bases and equal altitudes are equivalent.*

529. Cor. II. 1. *Two pyramids having equal altitudes are to each other as their bases.*

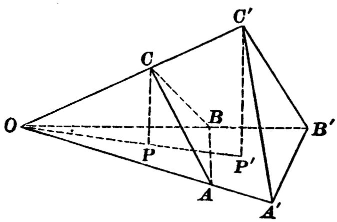
2. *Two pyramids having equivalent bases are to each other as their altitudes.*

3. *Any two pyramids are to each other as the products of their bases by their altitudes.*

530. Sch. The volume of any polyedron may be obtained by dividing it into pyramids.

Proposition XX. Theorem.

531. *Two tetraedrons having a triedral of one equal to a triedral of the other, are to each other as the products of the edges including the equal triedrals.*



Let V and V' denote the volumes of the tetraedrons $O\text{–}ABC$ and $O\text{–}A'B'C'$, having the common triedral O.

To prove $$\frac{V}{V'} = \frac{OA \times OB \times OC}{OA' \times OB' \times OC'}.$$

Draw CP and $C'P'$ perpendicular to the face $OA'B'$.

Let their plane intersect $OA'B'$ in the line OPP'.

Now, OAB and $OA'B'$ are the bases, and CP and $C'P$, the altitudes, of the triangular pyramids $C\text{–}OAB$ and $C'\text{–}OA'B'$.

Whence, $$\frac{V}{V'} = \frac{OAB \times CP}{OA'B' \times C'P'} \qquad (\S\ 529,\ 3.)$$

$$= \frac{OAB}{OA'B'} \times \frac{CP}{C'P'}. \qquad (1)$$

But, $$\frac{OAB}{OA'B'} = \frac{OA \times OB}{OA' \times OB'}. \qquad (\S\ 322.)$$

Also, the right triangles OCP and $OC'P'$ are similar.

(§ 257.)

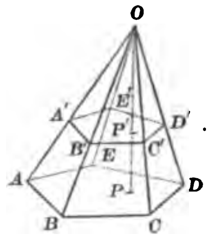
Whence, $$\frac{CP}{C'P'} = \frac{OC}{OC'}.$$

Substituting these values in (1), we have

$$\frac{V}{V'} = \frac{OA \times OB}{OA' \times OB'} \times \frac{OC}{OC'} = \frac{OA \times OB \times OC}{OA' \times OB' \times OC'}.$$

Proposition XXI. Theorem.

532. *The volume of a frustum of a pyramid is equal to the sum of its bases and a mean proportional between its bases, multiplied by one-third its altitude.*



Let AD' be a frustum of any pyramid $O\text{–}ABCDE$.

Denote the area of the lower base by B, the area of the upper base by b, and the altitude by H.

To prove vol. $AD' = (B + b + \sqrt{B \times b}) \times \frac{1}{3} H$. (§ 232.)

Draw the altitude OP, cutting $A'B'C'D'E'$ at P'.

Now, vol. $AD' =$ vol. $O\text{–}ABCDE -$ vol. $O\text{–}A'B'C'D'E'$

$$= B \times \tfrac{1}{3}\, OP - b \times \tfrac{1}{3}\, OP' \qquad (\S\ 527.)$$

$$= B \times \tfrac{1}{3}\, (H + OP') - b \times \tfrac{1}{3}\, OP'$$

$$= B \times \tfrac{1}{3}\, H + (B - b) \times \tfrac{1}{3}\, OP'. \qquad (1)$$

But, $$B : b = \overline{OP}^2 : \overline{OP'}^2. \qquad (\S\ 520.)$$

Taking the square root of each term, we have

$$\sqrt{B} : \sqrt{b} = OP : OP'. \qquad (\S\ 242.)$$

Then, $$\sqrt{B} - \sqrt{b} : \sqrt{b} = OP - OP' : OP' \qquad (\S\ 237.)$$

$$= H : OP'.$$

Whence, $$(\sqrt{B} - \sqrt{b}) \times OP' = \sqrt{b} \times H. \qquad (\S\ 231.)$$

Multiplying both members by $\sqrt{B} + \sqrt{b}$,

$$(B - b) \times OP' = (\sqrt{B \times b} + b) \times H.$$

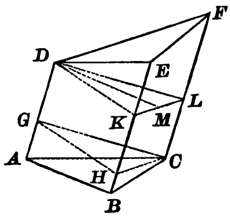
Substituting in (1), we have

$$\text{vol. } AD' = B \times \tfrac{1}{3}\, H + (\sqrt{B \times b} + b) \times \tfrac{1}{3}\, H$$

$$= (B + b + \sqrt{B \times b}) \times \tfrac{1}{3}\, H,$$

Proposition XXII. Theorem.

533. *The volume of a truncated triangular prism is equal to the product of a right section by one-third the sum of its lateral edges.*



Let GHC and DKL be right sections of the truncated triangular prism $ABC\text{–}DEF$.

To prove

$$\text{vol. } ABC\text{–}DEF = GHC \times \tfrac{1}{3}(AD + BE + CF).$$

Draw DM perpendicular to KL.

The truncated prism is composed of the triangular prism $GHC\text{–}DKL$, and the pyramids $D\text{–}EKLF$ and $C\text{–}ABHG$.

Now since the lateral edges of a prism are equal,

$$\text{vol. } GHC\text{–}DKL = GHC \times GD \qquad (\S\ 506.)$$

$$= GHC \times \tfrac{1}{3}(GD + HK + CL). \qquad (1)$$

Again, DM is the altitude of the pyramid $D\text{–}EKLF$. (§ 440.)

Whence, $\text{vol. } D\text{–}EKLF = EKLF \times \tfrac{1}{3} DM$ (§ 527.)

$= \tfrac{1}{2}(KE + LF) \times KL \times \tfrac{1}{3} DM$ (§ 317.)

$= \tfrac{1}{2} KL \times DM \times \tfrac{1}{3}(KE + LF)$.

But, $\tfrac{1}{2} KL \times DM = \text{area } DKL = \text{area } GHC$. (§ 313.)

Hence, $\text{vol. } D\text{–}EKLF = GHC \times \tfrac{1}{3}(KE + LF)$. (2)

In like manner, we may prove

$$\text{vol. } C\text{–}ABHG = GHC \times \tfrac{1}{3}(AG + BH). \qquad (3)$$

Adding (1), (2), and (3), we have

vol. $ABC\text{–}DEF$

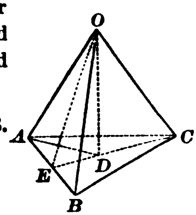
$$= GHC \times \tfrac{1}{3}(\overline{AG + GD} + \overline{BH + HK + KE} + \overline{CL + LF})$$

$$= GHC \times \tfrac{1}{3}(AD + BE + CF).$$

234. Cor. *The volume of a truncated right triangular prism is equal to the product of its base by one-third the sum of its lateral edges.*

EXERCISES.

17. Each side of the base of a regular triangular pyramid is 6, and its altitude is 4. Find its lateral edge, lateral area, and volume.



Let D be the centre of the base of the regular triangular pyramid $O\text{–}ABC$, and draw OD and AD; also, draw CDE perpendicular to AB, and join OE.

Now, $AD = AB \div \sqrt{3}$ (§ 357) $= \dfrac{6}{\sqrt{3}} = 2\sqrt{3}$.

Then, lat. edge OA

$= \sqrt{\overline{OD}^2 + \overline{AD}^2} = \sqrt{16 + 12} = \sqrt{28} = 2\sqrt{7}$.

The slant ht. $OE = \sqrt{\overline{OA}^2 - \overline{AE}^2} = \sqrt{28 - 9} = \sqrt{19}$.

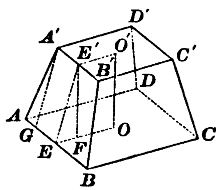
Then, lat. area $O\text{–}ABC = 9\sqrt{19}$ (§ 523).

Again, $CE = \sqrt{\overline{BC}^2 - \overline{BE}^2} = \sqrt{36 - 9} = \sqrt{27} = 3\sqrt{3}$.

Then, area $ABC = \frac{1}{2} \times 6 \times 3\sqrt{3}$ (§ 313) $= 9\sqrt{3}$.

Whence, vol. $O\text{–}ABC = \frac{1}{3} \times 9\sqrt{3} \times 4$ (§ 526) $= 12\sqrt{3}$.

18. Find the lateral edge, lateral area, and volume of a frustum of a regular quadrangular pyramid, the sides of whose bases are 17 and 7, and whose altitude is 12.



Let O and O' be the centres of the bases of the frustum of a regular quadrangular pyramid AC'.

Draw OE and $O'E'$ perpendicular to AB and $A'B'$, and $E'F$ and $A'G$ perpendicular to OE and AB; also, draw OO' and EE'.

Now, $EF = OE - O'E' = 8\frac{1}{2} - 3\frac{1}{2} = 5$.

Then, slant ht. $EE' = \sqrt{\overline{EF}^2 + \overline{E'F}^2} = \sqrt{25 + 144} = \sqrt{169} = 13$.

Whence, lat. area $AC' = \frac{1}{2}(68 + 28) \times 13$ (§ 524) $= 624$.

Again, $AG = AE - A'E' = 8\frac{1}{2} - 3\frac{1}{2} = 5$; and $A'G = EE' = 13$.

Whence, lat. edge $AA' = \sqrt{\overline{AG}^2 + \overline{A'G}^2} = \sqrt{25 + 169} = \sqrt{194}$.

Again, area $AC = 17^2 = 289$, and area $A'C' = 7^2 = 49$.

Then, a mean proportional between the areas of the bases

$$= \sqrt{17^2 \times 7^2} = 17 \times 7 = 119.$$

Whence, vol. $AC' = (289 + 49 + 119) \times 4$ (§ 532) $= 1828$.

Find the lateral edge, lateral area, and volume

19. Of a regular triangular pyramid, each side of whose base is 12, and whose altitude is 15.

20. Of a regular quadrangular pyramid, each side of whose base is 3, and whose altitude is 5.

21. Of a regular hexagonal pyramid, each side of whose base is 4, and whose altitude is 9.

22. Of a frustum of a regular triangular pyramid, the sides of whose bases are 18 and 6, and whose altitude is 24.

23. Of a frustum of a regular quadrangular pyramid, the sides of whose bases are 9 and 5, and whose altitude is 10.

24. Of a frustum of a regular hexagonal pyramid, the sides of whose bases are 8 and 4, and whose altitude is 12.

25. Find the volume of a truncated right triangular prism, the sides of whose base are 5, 12, and 13, and whose lateral edges are 3, 7, and 5, respectively.

26. Find the volume of a truncated right quadrangular prism, each side of whose base is 8, and whose lateral edges, taken in order, are 2, 6, 8, and 4, respectively.

27. The volume of a cube is $4\frac{17}{27}$ cu. ft. What is the area of its entire surface in square inches?

28. A box made of 2 in. plank, without a cover, measures on the outside 3 ft. 2 in. long, 2 ft. 3 in. wide, and 1 ft. 6 in. deep. How many cubic feet of material were used in its construction?

29. The volume of a right prism is 2310, and its base is a right triangle whose legs are 20 and 21. Find its lateral area.

30. Find the lateral area and volume of a right triangular prism, the sides of whose base are 4, 7, and 9, and whose altitude is 8.

31. Find the volume of a truncated right triangular prism, whose lateral edges are 11, 14, and 17, having for its base an isosceles triangle whose sides are 10, 13, and 13.

32. The altitude of a pyramid is 12 in., and its base is a square 9 in. on a side. What is the area of a section parallel to the base, whose distance from the vertex is 8 in.?

33. The altitude of a pyramid is 20 in., and its base is a rectangle whose dimensions are 10 in. and 15 in. What is the distance from the vertex of a section parallel to the base, whose area is 54 sq. in.?

34. The diagonal of a cube is $8\sqrt{3}$. Find its volume, and the area of its entire surface.

35. A trench is 124 ft. long, $2\frac{1}{2}$ ft. deep, 6 ft. wide at the top, and 5 ft. wide at the bottom. How many cubic feet of water will it contain?

36. The volume of a regular triangular prism is $96\sqrt{3}$, and one side of its base is 8. Find its lateral area.

37. The lateral area and volume of a regular hexagonal prism are 60 and $15\sqrt{3}$, respectively. Find its altitude, and one side of its base.

38. The slant height and lateral edge of a regular quadrangular pyramid are 25 and $\sqrt{674}$, respectively. Find its lateral area and volume.

39. The altitude and slant height of a regular hexagonal pyramid are 15 and 17, respectively. Find its lateral edge and volume.

40. The lateral edge of a frustum of a regular hexagonal pyramid is 10, and the sides of its bases are 10 and 4, respectively. Find its lateral area and volume.

41. Find the lateral area and volume of a regular quadrangular pyramid, the area of whose base is 100, and whose lateral edge is 13.

42. Find the lateral area and volume of a frustum of a regular triangular pyramid, the sides of whose bases are 12 and 6, and whose lateral edge is 5.

43. The lateral edges of a frustum of a quadrangular pyramid are equal; and its bases are rectangles, whose sides are 27 and 15, and 9 and 5, respectively. If the altitude of the frustum is 12, find its lateral area and volume.

44. Any straight line drawn through the centre of a parallelopiped, terminating in a pair of opposite faces, is bisected at that point.

45. The lateral surface of a pyramid is greater than its base.

46. The volume of a regular prism is equal to its lateral area, multiplied by one-half the apothem of its base.

47. The volume of a regular pyramid is equal to its lateral area, multiplied by one-third the distance from the centre of its base to any lateral face.

48. If E, F, G, and H are the middle points of the edges AB, AD, CD, and BC, respectively, of the tetraedron $ABCD$, prove that $EFGH$ is a parallelogram.

49. Find the area of the base of a regular quadrangular pyramid, whose lateral faces are equilateral triangles, and whose altitude is 5.

50. Two tetraedrons are equal if two faces and the included diedral of one are equal, respectively, to two faces and the included diedral of the other, if the equal parts are similarly placed.

51. Two tetraedrons are equal if three faces of one are equal, respectively, to three faces of the other, if the equal parts are similarly placed.

52. Find the area of the entire surface and the volume of a triangular pyramid, each of whose edges is 2.

53. The areas of the bases of a frustum of a pyramid are 12 and 75, and its altitude is 9. What is the altitude of the pyramid?

54. The sum of two opposite lateral edges of a truncated parallelopiped is equal to the sum of the other two lateral edges.

55. The volume of a truncated parallelopiped is equal to the area of a right section, multiplied by one-fourth the sum of the lateral edges.

56. A plane passed through the centre of a parallelopiped divided it into two equivalent solids.

57. If $ABCD$ is a rectangle, and EF any straight line not in its plane parallel to AB, the volume of the solid bounded by the figures $ABCD$, $ABFE$, $CDEF$, ADE, and BCF, is equal to

$$\tfrac{1}{6}\, h \times AD \times (2\,AB + EF),$$

where h is the perpendicular from E to $ABCD$. (§ 533.)

58. If $ABCD$ and $EFGH$ are rectangles lying in parallel planes, AB and BC being parallel to EF and FG, respectively, the solid bounded by the figures $ABCD$, $EFGH$, $ABFE$, $BCGF$, $CDHG$, and $DAEH$, is called a *rectangular prismoid.*

$ABCD$ and $EFGH$ are called the *bases* of the rectangular prismoid, and the perpendicular distance between them the *altitude.*

Prove that the volume of a rectangular prismoid is equal to the sum of its bases, plus four times a section equidistant from the bases, multiplied by one-sixth the altitude. (Ex. 57.)

59. Find the volume of a rectangular prismoid, the sides of whose bases are 10 and 7, and 6 and 5, respectively, and whose altitude is 9.

60. The volume of a triangular prism is equal to a lateral face, multiplied by one-half its perpendicular distance from any point in the opposite lateral edge.

61. The volume of a truncated right parallelopiped is equal to the area of its lower base, multiplied by the perpendicular drawn to the lower base from the centre of the upper base.

62. The perpendicular drawn to the lower base of a truncated right triangular prism from the intersection of the medians of the upper base, is equal to one-third the sum of the lateral edges.

63. The three planes passing through the lateral edges of a triangular pyramid, bisecting the sides of the base, meet in a common straight line.

64. A frustum of any pyramid is equivalent to the sum of three pyramids, having for their common altitude the altitude of the frustum, and for their bases the lower base, the upper base, and a mean proportional between the bases, of the frustum.

65. A monument is in the form of a frustum of a regular quadrangular pyramid 8 ft. in height, the sides of whose bases are 3 ft. and 2 ft., respectively, surmounted by a regular quadrangular pyramid 2 ft. in height. What is its weight, at 180 lb. to the cubic foot?

66. The altitude and lateral edge of a frustum of a regular triangular pyramid are 8 and 10, respectively, and each side of its upper base is $2\sqrt{3}$. Find its volume and lateral area.

67. A railway embankment, 1620 ft. in length, is $8\frac{1}{4}$ ft. wide at the top, $21\frac{1}{2}$ ft. wide at the bottom, and 6 ft. 4 in. high. How many cubic yards of earthwork does it contain?

68. The sides of the base, AB, BC, and CA, of a truncated right triangular prism $ABC\text{-}DEF$, are 15, 4, and 12, respectively, and the lateral edges, AD, BE, and CF, are 15, 7, and 10, respectively. Find the area of the upper base, DEF.

69. If $ABCD$ is a tetraedron, the section made by a plane parallel to each of the edges AB and CD is a parallelogram. (§ 415.)

70. In a tetraedron $ABCD$, a plane is drawn through the edge CD perpendicular to AB, intersecting the faces ABC and ABD in CE and ED. If the bisector of the angle CED meets CD at F, prove

$$CF : DF = \text{area } ABC : \text{area } ABD.$$

71. The sum of the squares of the four diagonals of any parallelopiped is equal to the sum of the squares of its twelve edges. (Ex. 75, p. 226.)

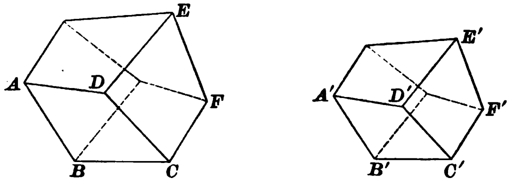
72. If the four diagonals of a quadrangular prism pass through a common point, the prism is a parallelopiped.

SIMILAR POLYEDRONS.

535. Def. Two polyedrons are said to be *similar* when they are bounded by the same number of faces, similar each to each and similarly placed, and have their homologous polyedrals equal.

Proposition XXIII. Theorem.

536. *The ratio of any two homologous edges of two similar polyedrons is equal to the ratio of any other two homologous edges.*



In the similar polyedrons AF and $A'F'$, let the edges AB and EF be homologous to the edges $A'B'$ and $E'F'$.

To prove $$\frac{AB}{A'B'} = \frac{EF}{E'F'}.$$

The face AC is similar to $A'C'$, and DF to $D'F'$. (§ 535.)

Whence, $$\frac{AB}{A'B'} = \frac{CD}{C'D'} = \frac{EF}{E'F'}. \qquad (§ 253, I.)$$

537. Cor. I. We have

$$\frac{\text{area } ABCD}{\text{area } A'B'C'D'} = \frac{\overline{AB}^2}{\overline{A'B'}^2} \qquad (§ 323.)$$

$$= \frac{\overline{EF}^2}{\overline{E'F'}^2}. \qquad (§ 536.)$$

Therefore, *any two homologous faces of two similar polyedrons are to each other as the squares of any two homologous edges.*

538. Cor. II. We have

$$\frac{ABCD}{A'B'C'D'} = \frac{CDEF}{C'D'E'F'} = \text{etc.} = \frac{\overline{AB}^2}{\overline{A'B'}^2}. \quad (\S\ 537.)$$

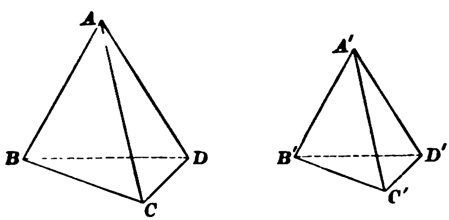
Whence,

$$\frac{ABCD + CDEF + \text{etc.}}{A'B'C'D' + C'D'E'F' + \text{etc.}} = \frac{\overline{AB}^2}{\overline{A'B'}^2}. \quad (\S\ 239.)$$

Hence, *the entire surfaces of two similar polyedrons are to each other as the squares of any two homologous edges.*

Proposition XXIV. Theorem.

539. *Two tetraedrons are similar when the faces including a triedral of one are similar to the faces including a triedral of the other, and similarly placed.*



In the tetraedrons $ABCD$ and $A'B'C'D'$, let the faces ABC, ACD, and ADB be similar to the faces $A'B'C'$, $A'C'D'$, and $A'D'B'$, respectively.

To prove $ABCD$ and $A'B'C'D'$ similar.

From the given similar faces, we have

$$\frac{BC}{B'C'} = \frac{AC}{A'C'} = \frac{CD}{C'D'} = \frac{AD}{A'D'} = \frac{BD}{B'D'}.$$

Hence, the faces BCD and $B'C'D'$ are similar. (§ 259.)

Again, since the angles BAC, CAD, and DAB are equal respectively to the angles $B'A'C'$, $C'A'D'$, and $D'A'B'$, the triedrals $A\text{–}BCD$ and $A'\text{–}B'C'D'$ are equal. (§ 464, I.)

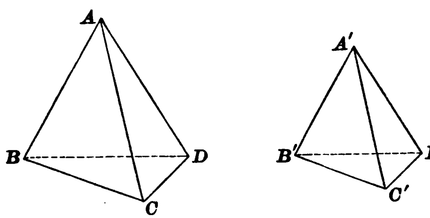
In like manner, any two homologous triedrals are equal.

Hence, $ABCD$ and $A'B'C'D'$ are similar. (§ 535.)

540. Cor. *If a tetraedron be cut by a plane parallel to one of its faces, the tetraedron cut off is similar to the given tetraedron.*

Proposition XXV. Theorem.

541. *Two tetraedrons are similar when a diedral of one is equal to a diedral of the other, and the faces including the equal diedrals are similar each to each, and similarly placed.*



In the tetraedrons $ABCD$ and $A'B'C'D'$, let the diedral AB be equal to $A'B'$; and let the faces ABC and ABD be similar to $A'B'C'$ and $A'B'D'$, respectively.

To prove $ABCD$ and $A'B'C'D'$ similar.

Let the tetraedron $A'B'C'D'$ be applied to $ABCD$, so that the diedral $A'B'$ shall coincide with its equal AB, the point A' falling at A.

Then since $\angle B'A'C' = \angle BAC$, and $\angle B'A'D' = \angle BAD$, the edge $A'C'$ will coincide with AC, and $A'D'$ with AD.

Therefore, $\angle C'A'D' = \angle CAD$.

Again, from the given similar faces, we have

$$\frac{A'C}{AC} = \frac{A'B'}{AB} = \frac{A'D'}{AD}.$$

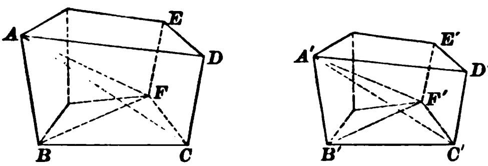
Hence, the triangle $C'A'D'$ is similar to CAD. (§ 260.)

Then, the faces including the triedral $A'\text{–}B'C'D'$ are similar to the faces including the triedral $A\text{–}BCD$, and similarly placed.

Therefore, $ABCD$ and $A'B'C'D'$ are similar. (§ 539.)

PROPOSITION XXVI. THEOREM.

542. *Two similar polyedrons may be decomposed into the same number of tetraedrons, similar each to each, and similarly placed.*



Let AF and $A'F'$ be two similar polyedrons, the vertices A and A' being homologous.

To prove that they may be decomposed into the same number of tetraedrons, similar each to each, and similarly placed.

Divide all the faces of AF, except those having A as a vertex, into triangles; and draw straight lines from A to their vertices.

In like manner, divide all the faces of $A'F'$, except those having A' as a vertex, into triangles similar to those in AF, and similarly placed. (§ 267.)

Draw straight lines from A' to their vertices.

The given polyedrons are then decomposed into the same number of tetraedrons, similarly placed.

Let $ABCF$ and $A'B'C'F'$ be two homologous tetraedrons.

The triangles ABC and BCF are similar to $A'B'C'$ and $B'C'F'$, respectively. (§ 267.)

And since the given polyedrons are similar, the homologous diedrals BC and $B'C'$ are equal.

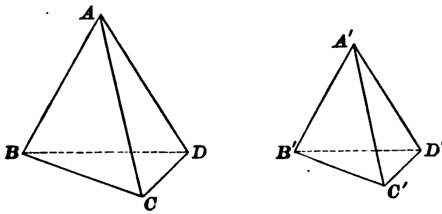
Therefore, $ABCF$ and $A'B'C'F'$ are similar. (§ 541.)

In like manner, we may prove any two homologous tetraedrons similar.

Hence, the given polyedrons are decomposed into the same number of tetraedrons, similar each to each, and similarly placed.

PROPOSITION XXVII. THEOREM.

543. *Two similar tetraedrons are to each other as the cubes of their homologous edges.*



Let V and V' denote the volumes of the similar tetraedrons $ABCD$ and $A'B'C'D'$, the vertices A and A' being homologous.

To prove $$\frac{V}{V'} = \frac{\overline{AB}^3}{\overline{A'B'}^3}.$$

Since the triedrals at A and A' are equal, we have

$$\frac{V}{V'} = \frac{AB \times AC \times AD}{A'B' \times A'C' \times A'D'} \qquad (\S\ 531.)$$

$$= \frac{AB}{A'B'} \times \frac{AC}{A'C'} \times \frac{AD}{A'D'}. \qquad (1)$$

But, $$\frac{AC}{A'C'} = \frac{AD}{A'D'} = \frac{AB}{A'B'}. \qquad (\S\ 536.)$$

Substituting in (1), we have

$$\frac{V}{V'} = \frac{AB}{A'B'} \times \frac{AB}{A'B'} \times \frac{AB}{A'B'} = \frac{\overline{AB}^3}{\overline{A'B'}^3}.$$

544. COR. *Any two similar polyedrons are to each other as the cubes of their homologous edges.*

For any two similar polyedrons may be decomposed into the same number of tetraedrons, similar each to each (§ 542).

And any two homologous tetraedrons are to each other as the cubes of their homologous edges (§ 543), or as the cubes of any two homologous edges of the polyedrons (§ 536).

REGULAR POLYEDRONS.

545. Def. A *regular polyedron* is a polyedron whose faces are equal regular polygons, and whose polyedrals are all equal.

Proposition XXVIII. Theorem.

546. *Not more than five regular convex polyedrons are possible.*

A convex polyedral must have at least three faces, and the sum of its face angles must be less than 360° (§ 463).

1. *With equilateral triangles.*

Since the angle of an equilateral triangle is 60°, we may form a convex polyedral by combining either 3, 4, or 5 equilateral triangles.

Not more than 5 equilateral triangles can be combined to form a convex polyedral. (§ 463.)

Hence, not more than three regular convex polyedrons can be formed with equilateral triangles.

2. *With squares.*

Since the angle of a square is 90°, we may form a convex polyedral by combining 3 squares.

Not more than 3 squares can be combined to form a convex polyedral.

Hence, not more than one regular convex polyedron can be formed with squares.

3. *With regular pentagons.*

Since the angle of a regular pentagon is 108°, we may form a convex polyedral by combining 3 regular pentagons.

Not more than 3 regular pentagons can be combined to form a convex polyedral.

Hence, not more than one regular convex polyedron can be formed with regular pentagons.

Since the angle of a regular hexagon is 120°, no convex polyedral can be formed by combining regular hexagons.

In like manner, no convex polyedral can be formed by combining regular polygons of more than six sides.

Hence, not more than five regular convex polyedrons are possible.

Proposition XXIX. Problem.

547. *With a given edge, to construct a regular polyedron.*

We will now prove, by actual construction, that five regular convex polyedrons are possible:

1. The regular tetraedron, bounded by 4 equilateral triangles.

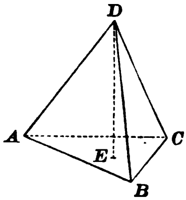
2. The regular hexaedron, or cube, bounded by 6 squares.

3. The regular octaedron, bounded by 8 equilateral triangles.

4. The regular dodecaedron, bounded by 12 regular pentagons.

5. The regular icosaedron, bounded by 20 equilateral triangles.

1. *To construct a regular tetraedron.*



Let AB be the given edge.

Construct the equilateral triangle ABC.

At its centre E, draw ED perpendicular to ABC; and take the point D so that $AD = AB$.

Draw AD, BD, and CD.

Then, $ABCD$ is a regular tetraedron.

For since A, B, and C are equally distant from E,

$$AD = BD = CD. \qquad (\S\ 407,\ \text{I.})$$

Hence, the six edges of the tetraedron are all equal.

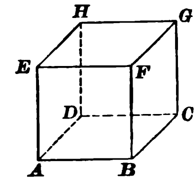
Then, the faces are equal equilateral triangles. (§ 69.)

And since the angles of the faces are all equal, the triedrals whose vertices are A, B, C, and D are equal.

(§ 464, I.)

Therefore, $ABCD$ is a regular tetraedron.

2. *To construct a regular hexaedron, or cube.*



Let AB be the given edge.

Construct the square $ABCD$.

Draw AE, BF, CG, and DH, each equal to AB and perpendicular to $ABCD$.

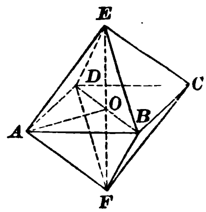
Draw EF, FG, GH, and HE.

Then, AG is a regular hexaedron.

For by construction, its faces are equal squares.

Hence, its triedrals are all equal. (§ 464, I.)

3. *To construct a regular octaedron.*



Let AB be the given edge.

Construct the square $ABCD$; through its centre O draw EOF perpendicular to $ABCD$, making $OE = OF = OA$.

Join the points E and F to A, B, C, and D.

Then, $AEFC$ is a regular octaedron.

For draw OA, OB, and OD.

Then in the right triangles AOB, AOE, and AOF,

$$OA = OB = OE = OF.$$

Therefore, $\triangle AOB = \triangle AOE = \triangle AOF$. (§ 63.)

Whence, $AB = AE = AF$. (§ 66.)

Then the eight edges terminating at E and F are all equal. (§ 407, I.)

Thus, the twelve edges of the octaedron are all equal, and the faces are equal equilateral triangles. (§ 69.)

Again, by construction, the diagonals of the quadrilateral $BEDF$ are equal, and bisect each other at right angles.

Hence, $BEDF$ is a square equal to $ABCD$, and OA is perpendicular to its plane. (§ 400.)

Therefore, the pyramids A–$BEDF$ and E–$ABCD$ are equal; and hence the polyedrals A–$BEDF$ and E–$ABCD$ are equal.

In like manner, any two polyedrals are equal.

Hence, $AEFC$ is a regular octaedron.

4. *To construct a regular dodecaedron.*

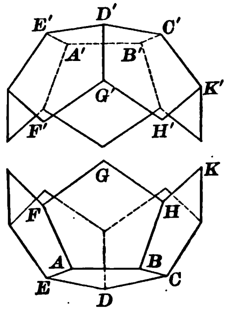


Fig. 1.

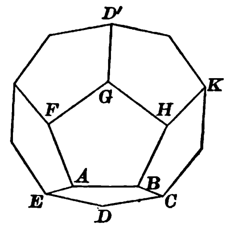


Fig. 2.

Let AB be the given edge.

Construct the regular pentagon $ABCDE$ (Fig. 1).

To $ABCDE$ join five equal regular pentagons, so inclined as to form equal triedrals at the vertices A, B, C, D, and E. (§ 464, I.)

Then there is formed a convex surface AK, composed of six regular pentagons, as shown in the lower portion of Fig. 1.

Construct a second surface $A'K'$ equal to AK, as shown in the upper portion of Fig. 1.

The surfaces AK and $A'K'$ may be combined as shown in Fig. 2, so as to form at F a triedral equal to that at A, having for its faces the regular pentagons about the vertices F and F' in Fig. 1. (§ 464, I.)

Then, AK is a regular dodecaedron.

For since G' falls at G, and the diedral FG and the face angles FGH and FGD' (Fig. 2) are equal respectively to the diedral and face angles of the triedral F, the faces about the vertex G will form a triedral equal to that at F.

Continuing in this way, it may be proved that at each of the vertices H, K, etc., there is formed a triedral equal to that at F.

Therefore, AK is a regular dodecaedron.

5. *To construct a regular icosaedron.*

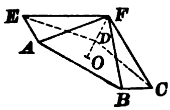


Fig. 1.

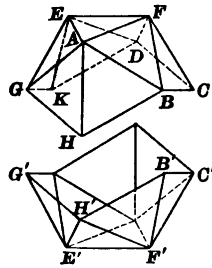


Fig. 2.

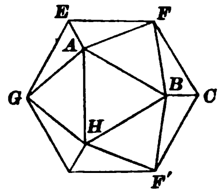


Fig. 3.

Let AB be the given edge.

Construct the regular pentagon $ABCDE$ (Fig. 1).

At its centre O draw OF perpendicular to $ABCDE$, making $AF = AB$.

Draw AF, BF, CF, DF, and EF.

Then $F\text{–}ABCDE$ is a regular pyramid whose lateral faces are equal equilateral triangles. (§ 69.)

Construct two other regular pyramids, $A\text{–}BFEGH$ and $E\text{–}AFDKG$, each equal to $F\text{–}ABCDE$.

Place them as shown in the upper portion of Fig. 2, so that the faces ABF and AEF of $A\text{–}BFEGH$, and the faces AEF and DEF of $E\text{–}AFDKG$, shall coincide with the corresponding faces of $F\text{–}ABCDE$.

Then there is formed a convex surface GC, composed of ten equilateral triangles.

Construct a second surface $G'C'$ equal to GC, as shown in the lower portion of Fig. 2.

Then the surfaces GC and $G'C'$ may be combined as shown in Fig. 3, so that the edges GH and HB shall coincide with $G'H'$ and $H'B'$, respectively.

For since the diedrals AH, $E'H'$, and $F'H'$ are equal to the diedrals of the polyedral F, the faces about the vertices H and H' may be made to form a polyedral at H equal to that at F. (§ 459.)

Then since the diedrals *FB*, *AB*, *HB*, and *F′B* (Fig. 3) are equal to the diedrals of the polyedral *F*, the faces about the vertex *B* will form a polyedral equal to that at *F*.

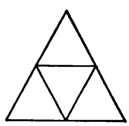
Continuing in this way, it may be shown that at each of the vertices *C*, *D*, etc., there is formed a polyedral equal to that at *F*.

Therefore, *GC* is a regular icosaedron.

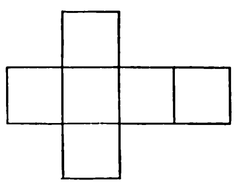
548. Sch. To construct the regular polyedrons, draw the following diagrams accurately on cardboard.

Cut the figures out entire, and on the interior lines cut the cardboard half through.

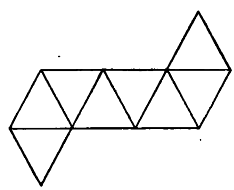
The edges may then be brought together so as to form the respective solids.



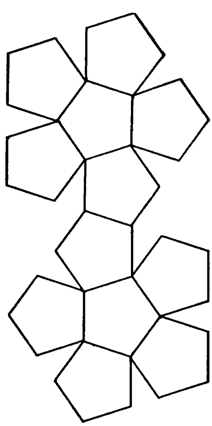
Tetraedron.



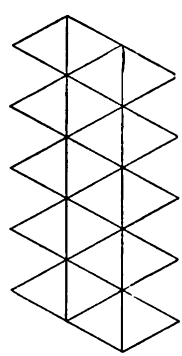
Hexaedron.



Octaedron.



Dodecaedron.



Icosaedron.

EXERCISES.

73. If the volume of a pyramid whose altitude is 7 in. is 686 cu. in., what is the volume of a similar pyramid whose altitude is 12 in.?

74. If the volume of a prism whose altitude is 9 ft. is 171 cu. ft., what is the altitude of a similar prism whose volume is $50\frac{2}{3}$ cu. ft.?

75. Two bins of similar form contain, respectively, 375 and 648 bushels of wheat. If the first bin is 3 ft. 9 in. long, what is the length of the second?

76. A pyramid whose altitude is 10 in., weighs 24 lb. At what distance from its vertex must it be cut by a plane parallel to its base so that the frustum cut off may weigh 12 lb.?

77. An edge of a polyedron is 56, and the homologous edge of a similar polyedron is 21. The area of the entire surface of the second polyedron is 135, and its volume is 162. Find the area of the entire surface, and the volume, of the first polyedron.

78. The area of the entire surface of a tetraedron is 147, and its volume is 686. If the area of the entire surface of a similar tetraedron is 48, what is its volume?

79. The area of the entire surface of a tetraedron is 75, and its volume is 500. If the volume of a similar tetraedron is 32, what is the area of its entire surface?

80. The homologous edges of three similar tetraedrons are 3, 4, and 5, respectively. Find the homologous edge of a similar tetraedron equivalent to their sum.

81. State and prove the converse of Prop. XXVI.

82. The volume of a regular tetraedron is equal to the cube of its edge multiplied by $\frac{1}{12}\sqrt{2}$.

83. The volume of a regular octaedron is equal to the cube of its edge multiplied by $\frac{1}{3}\sqrt{2}$.

84. The volume of a regular tetraedron is $18\sqrt{2}$. Find the area of its entire surface.

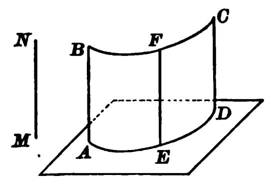
85. The sum of the perpendiculars drawn to the faces from any point within a regular tetraedron is equal to the altitude of the tetraedron.

BOOK VIII.

THE CYLINDER, CONE, AND SPHERE.

DEFINITIONS.

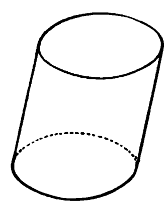
549. A *cylindrical surface* is a surface generated by a moving straight line, which constantly intersects a given curve, and in all of its positions is parallel to a given straight line, not in the plane of the curve.



Thus, if the line AB moves so as to constantly intersect the curve AD, and is constantly parallel to the line MN, not in the plane of the curve, it generates a cylindrical surface.

550. The moving straight line is called the *generatrix*, and the curve the *directrix*; any position of the generatrix, as EF, is called an *element* of the surface.

551. A *cylinder* is a solid bounded by a cylindrical surface, and two parallel planes.



The parallel planes are called the *bases* of the cylinder, and the cylindrical surface the *lateral surface.*

The *altitude* of a cylinder is the perpendicular distance between the planes of its bases.

NOTE. We shall use the phrase "element of a cylinder" to signify an *element of its lateral surface.*

552. It follows from the definition of § 551 that

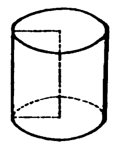
The elements of a cylinder are equal and parallel. (§ 418.)

553. A *right cylinder* is a cylinder whose elements are perpendicular to its bases.

A *circular cylinder* is a cylinder whose base is a circle.

The *axis* of a circular cylinder is a straight line drawn through the centre of its base parallel to its elements.

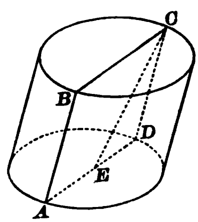
554. A right circular cylinder is called a *cylinder of revolution*; for it may be generated by the revolution of a rectangle about one of its sides as an axis.



555. *Similar* cylinders of revolution are cylinders generated by the revolution of similar rectangles about homologous sides as axes.

PROPOSITION I. THEOREM.

556. *A section of a cylinder made by a plane passing through an element is a parallelogram.*



Let $ABCD$ be a section of the cylinder AC, made by a plane passing through the element AB.

To prove $ABCD$ a parallelogram.

Draw CE in the plane $ABCD$ parallel to AB.

Then CE is an element of the lateral surface. (§ 552.)

Therefore, CE must be the intersection of the plane $ABCD$ with the lateral surface of the cylinder.

Hence, CE coincides with CD, and CD is parallel to AB.

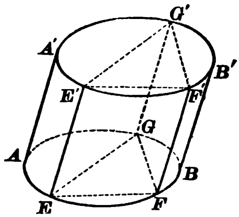
Again, AD is parallel to BC. (§ 417.)

Therefore, $ABCD$ is a parallelogram.

557. Cor. *A section of a right cylinder made by a plane passing through an element is a rectangle.*

Proposition II. Theorem.

558. *The bases of a cylinder are equal.*



Let AB' be a cylinder.

To prove its bases AB and $A'B'$ equal.

Let E', F', and G' be any three points in the perimeter of $A'B'$, and draw the elements EE', FF', and GG'.

Draw EF, FG, GE, $E'F'$, $F'G'$, and $G'E'$.

Now, EE' and FF' are equal and parallel. (§ 552.)

Therefore, $EE'F'F$ is a parallelogram. (§ 109.)

Hence, $E'F' = EF$. (§ 104.)

Similarly, $E'G' = EG$, and $F'G' = FG$.

Therefore, $\triangle E'F'G' = \triangle EFG$. (§ 69.)

Then the base $A'B'$ may be superposed upon AB so that the points E', F', and G' shall fall at E, F, and G.

But E' is any point in the perimeter of $A'B'$.

Hence, *every* point in the perimeter of $A'B'$ will fall in the perimeter of AB, and $A'B'$ is equal to AB.

559. Cor. I. *The sections of a cylinder made by two parallel planes cutting all its elements are equal.*

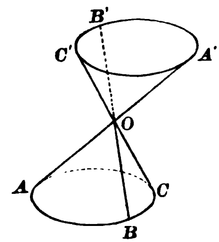
For they are the bases of a cylinder.

560. Cor. II. *A section of a cylinder made by a plane parallel to the base is equal to the base.*

THE CONE.

DEFINITIONS.

561. A *conical surface* is a surface generated by a moving straight line, which constantly intersects a given curve, and passes through a given point not in the plane of the curve.



Thus, if the line OA moves so as to constantly intersect the curve ABC, and constantly passes through the point O, not in the plane of the curve, it generates a conical surface.

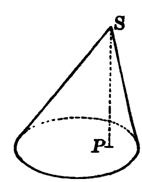
562. The moving straight line is called the *generatrix*, and the curve the *directrix*.

The given point is called the *vertex*; and any position of the generatrix, as OB, is called an *element* of the surface.

563. If the generatrix be supposed indefinite in length, it will generate two conical surfaces, $O\text{–}A'B'C'$ and $O\text{–}ABC$.

These are called the *upper* and *lower nappes*, respectively.

564. A *cone* is a solid bounded by a conical surface, and a plane cutting all its elements.



The plane is called the *base* of the cone, and the curved surface the *lateral surface.*

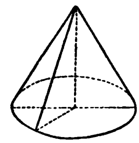
The *altitude* of a cone is the perpendicular distance from the vertex to the plane of the base.

565. A *circular cone* is a cone whose base is a circle.

The *axis* of a circular cone is a straight line drawn from the vertex to the centre of the base.

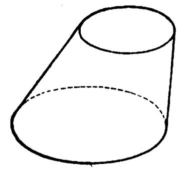
566. A *right circular cone* is a circular cone whose axis is perpendicular to its base.

567. A right circular cone is called a *cone of revolution*, for it may be generated by the revolution of a right triangle about one of its legs as an axis.



568. *Similar* cones of revolution are cones generated by the revolution of similar right triangles about homologous legs as axes.

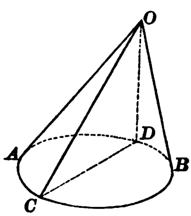
569. A *frustum of a cone* is that portion of a cone included between the base and a plane parallel to the base.



The *altitude* of a frustum is the perpendicular distance between the planes of its bases.

PROPOSITION III. THEOREM.

570. *A section of a cone made by a plane passing through the vertex is a triangle.*



Let OCD be a section of the cone OAB, made by a plane passing through the vertex O.

To prove OCD a triangle.

Draw straight lines in the plane OCD from O to the points C and D.

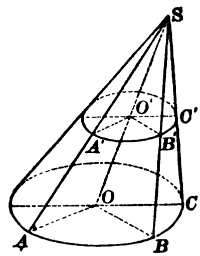
These lines are elements of the lateral surface. (§ 562.)

Then they must be the lines of intersection of the plane OCD with the lateral surface of the cone.

Therefore, OC and OD are straight lines, and OCD is a triangle.

PROPOSITION IV. THEOREM.

571. *A section of a circular cone made by a plane parallel to the base is a circle.*



Let $A'B'C'$ be a section of the circular cone $S\text{–}ABC$, made by a plane parallel to the base.

To prove $A'B'C'$ a circle.

Let the axis OS intersect the plane $A'B'C'$ at O'.

Let A' and B' be any two points in the perimeter $A'B'C'$.

Let the planes determined by these points and OS intersect the base in the radii OA and OB, and the lateral surface in the elements SA and SB. (§ 572.)

Then, $O'A'$ is parallel to OA, and $O'B'$ to OB. (§ 417.)

Therefore, the triangles $A'O'S$ and $B'O'S$ are similar to the triangles AOS and BOS. (§ 258.)

Whence, $$\frac{O'A'}{OA} = \frac{SO'}{SO}, \text{ and } \frac{O'B'}{OB} = \frac{SO'}{SO}.$$

Then, $$\frac{O'A'}{OA} = \frac{O'B'}{OB}.$$

But, $$OA = OB. \qquad (§ 143.)$$

Whence, $$O'A' = O'B'.$$

Now A' and B' are *any* two points in the perimeter $A'B'C'$.

Therefore, the section $A'B'C'$ is a circle.

572. COR. *The axis of a circular cone passes through the centre of every section parallel to the base.*

THE SPHERE.

DEFINITIONS.

573. A *sphere* is a solid bounded by a surface, all points of which are equally distant from a point within called the *centre.*

574. A *radius* of a sphere is a straight line drawn from the centre to the surface.

A *diameter* is a straight line drawn through the centre, having its extremities in the surface.

575. It follows from the definition of § 574 that

All radii of a sphere are equal.

Also, all its diameters are equal, since each is the sum of two radii.

576. A sphere may be generated by the revolution of a semicircle about its diameter as an axis.

577. *Two spheres are equal when their radii are equal.*

For they can evidently be applied one to the other so that their surfaces shall coincide throughout.

578. Conversely, *the radii of equal spheres are equal.*

579. A straight line or plane is said to be *tangent to a sphere* when it has but one point in common with the surface of the sphere.

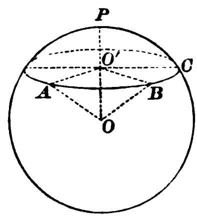
The common point is called the *point of contact,* or *point of tangency.*

580. A polyedron is said to be *inscribed in a sphere* when its vertices lie in the surface of the sphere; in this case the sphere is said to be circumscribed about the polyedron.

A polyedron is said to be *circumscribed about a sphere* when its faces are tangent to the sphere; in this case the sphere is said to be inscribed in the polyedron.

PROPOSITION V. THEOREM.

581. *A section of a sphere made by a plane is a circle.*



Let ABC be a section of the sphere APC made by a plane.

To prove ABC a circle.

Let O be the centre of the sphere.

Draw OO' perpendicular to the plane ABC.

Let A and B be any two points in the perimeter of ABC, and draw OA, OB, $O'A$, and $O'B$.

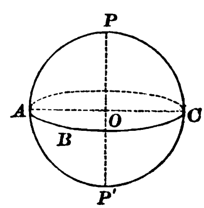
Now, $\qquad OA = OB. \qquad$ (§ 575.)

Whence, $\qquad O'A = O'B. \qquad$ (§ 408.)

But A and B are *any* two points in the perimeter of ABC.

Therefore, ABC is a circle.

582. DEF. A *great circle* of a sphere is a section made by a plane passing through the centre; as ABC.



A *small circle* is a section made by a plane which does not pass through the centre.

The diameter perpendicular to a circle of a sphere is called the *axis* of the circle, and its extremities are called the *poles*.

583. COR. I. *The axis of a circle of a sphere passes through the centre of the circle.*

584. Cor. II. *All great circles of a sphere are equal.*

For their radii are radii of the sphere.

585. Cor. III. *Every great circle bisects the sphere and its surface.*

For if the parts be separated, and placed so that their plane surfaces coincide, the spherical surfaces falling on the same side of this plane, the two spherical surfaces will coincide throughout; for all points of either are equally distant from the centre.

586. Cor. IV. *Any two great circles bisect each other.*

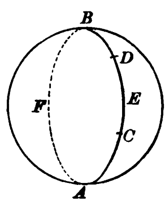
For the intersection of their planes passes through the centre of the sphere, and hence is a diameter of each circle.

587. Cor. V. *An arc of a great circle, less than a semi-circumference, may be drawn between any two points on the surface of a sphere, and but one.*

For the two points, together with the centre of the sphere, determine a plane which intersects the surface of the sphere in the arc required.

Note. If the points lie at the extremities of a diameter of the sphere, an indefinitely great number of arcs of great circles may be drawn between them; for an indefinitely great number of planes can be drawn through the diameter.

588. Def. The *distance* between two points on the surface of a sphere is the arc of a great circle, less than a semi-circumference, drawn between them.



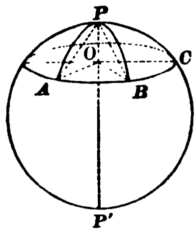
Thus, the distance between the points C and D is the arc CED, and not CFD.

589. Cor. VI. *An arc of a circle may be drawn through any three points on the surface of a sphere.*

For the three points determine a plane, which intersects the surface of the sphere in the arc required.

PROPOSITION VI. THEOREM.

590. ***All points in the circumference of a circle of a sphere are equally distant from each of its poles.***



Let P and P' be the poles of the circle ABC of the sphere APC.

To prove that all points in the circumference ABC are equally distant (§ 588) from P, and also from P'.

Let A and B be any two points in the circumference ABC, and draw the arcs of great circles PA and PB.

Draw the axis PP', intersecting the plane ABC at O.

Draw OA, OB, PA, and PB.

Now O is the centre of the circle ABC. (§ 583.)

Whence, $OA = OB$. (§ 143.)

Therefore, chord PA = chord PB. (§ 407, I.)

Whence, arc PA = arc PB. (§ 157.)

But A and B are *any* two points in the circumference ABC.

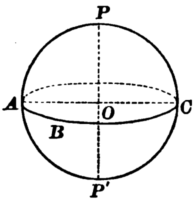
Hence, all points in the circumference ABC are equally distant from P.

In like manner, we may prove that all points in the circumference ABC are equally distant from P'.

591. DEF. The *polar distance* of a circle of a sphere is the distance (§ 588) from the nearer of its poles to the circumference of the circle.

Thus, the polar distance of the circle ABC is the arc PA.

592. Cor. *The polar distance of a great circle is a quadrant.*



Let PA be the polar distance of the great circle ABC.

To prove PA a quadrant (§ 146).

Let O be the centre of the sphere, and draw OA and OP.

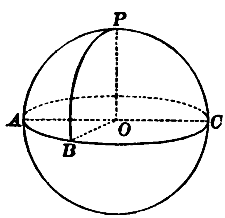
Then, $\angle POA$ is a right angle. (§ 398.)

Therefore, the arc PA is a quadrant. (§ 191.)

593. Sch. The term *quadrant,* in Spherical Geometry, usually signifies a quadrant of a great circle.

Proposition VII. Theorem.

594. *If a point on the surface of a sphere lies at a quadrant's distance from each of two points in the arc of a great circle, it is the pole of that arc.*



Let P be a point on the surface of the sphere AC, at a quadrant's distance from each of the points A and B.

To prove P the pole of the arc of a great circle AB.

Draw the radii OA, OB, and OP.

Then since the arcs PA and PB are quadrants, the angles POA and POB are right angles. (§ 191.)

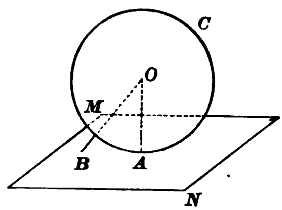
Then, PO is perpendicular to the plane OAB. (§ 400.)

Therefore, P is the pole of the arc AB.

NOTE. If the two points lie at the extremities of a diameter, the theorem is not necessarily true; for P is the pole of only one of the great circles which can be drawn through the points A and C.

PROPOSITION VIII. THEOREM.

595. *A plane perpendicular to a radius of a sphere at its extremity is tangent to the sphere.*



Let the plane MN be perpendicular to the radius OA of the sphere AC at its extremity A.

To prove MN tangent to the sphere.

Let B be any point of MN except A, and draw OB.

Then, $OB > OA$. (§ 402.)

Whence, B lies without the sphere.

Then every point of MN except A lies without the sphere, and MN is tangent to the sphere. (§ 579.)

596. COR. (Converse of Prop. VIII.) *A plane tangent to a sphere is perpendicular to the radius drawn to the point of contact.*

Let the plane MN be tangent to the sphere AC.

To prove that MN is perpendicular to the radius OA drawn to the point of contact.

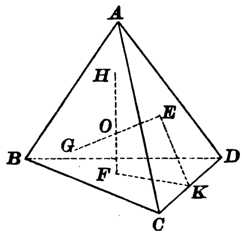
If MN is tangent to the sphere at A, every point of MN except A lies without the sphere.

Then OA is the shortest line that can be drawn from O to MN.

Whence, OA is perpendicular to MN. (§ 402.)

Proposition IX. Theorem.

597. *A sphere may be circumscribed about any tetraedron.*



Let $ABCD$ be any tetraedron.

To prove that a sphere may be circumscribed about it.

Draw EK in the face ACD, and FK in the face BCD, perpendicular to CD at its middle point K.

Let E and F be the centres of the circumscribed circles of the triangles ACD and BCD (§ 222).

Draw EG and FH perpendicular to the planes ACD and BCD.

Now the plane determined by EK and FK is perpendicular to CD. (§ 400.)

Therefore, this plane is perpendicular to each of the planes ACD and BCD. (§ 444.)

Then EG, being perpendicular to the plane ACD, lies in the plane determined by EK and FK. (§ 441.)

In like manner, FH lies in this plane.

Therefore, EG and FH must meet at some point, as O.

Since O is in the perpendicular EG, it is equally distant from A, C, and D. (§ 407, I.)

And since it is in the perpendicular FH, it is equally distant from B, C, and D.

Hence, O is equally distant from A, B, C, and D; and the sphere described with O as a centre, and OA as a radius, will be circumscribed about the tetraedron.

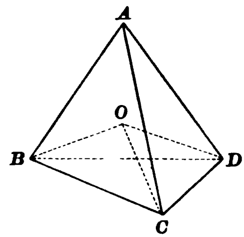
598. COR. I. *But one sphere can be circumscribed about a given tetraedron.*

For the centre of any circumscribed sphere must lie in the perpendicular drawn to each face at the centre of its circumscribed circle. (§ 408.)

599. COR. II. *The planes perpendicular to the edges of a tetraedron at their middle points meet in a common point.*

PROPOSITION X. THEOREM.

600. *A sphere may be inscribed in any tetraedron.*



Let $ABCD$ be any tetraedron.

To prove that a sphere may be inscribed in it.

Draw the planes OBC, OCD, and ODB, bisecting the diedrals $ABCD$, $ACDB$, and $ADBC$, respectively.

Then since O is in the plane OBC, it is equally distant from the faces ABC and BCD. (§ 446.)

In like manner, O is equally distant from the faces ACD and BCD, and from the faces ABD and BCD.

Hence, O is equally distant from the four faces of the tetraedron; and the sphere described with O as a centre, and the perpendicular from O to either face as a radius, will be inscribed in the tetraedron.

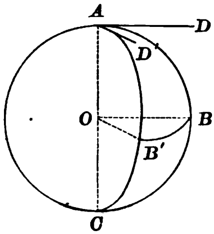
601. COR. *The planes bisecting the diedrals of a tetraedron meet in a common point.*

602. Definitions. The *angle* between two intersecting curves is the angle included between tangents to the curves at their common point.

A *spherical angle* is the angle between two intersecting arcs of great circles.

Proposition XI. Theorem.

603. *A spherical angle is measured by the arc of a great circle described with its vertex as a pole, included between its sides produced if necessary.*



Let ABC and $AB'C$ be arcs of great circles on the surface of the sphere AC whose centre is O.

Draw AD and AD' tangent to ABC and $AB'C$.

Then DAD' is the angle between the arcs ABC and $AB'C$. (§ 602.)

Draw the diameter AC; also, draw OB and OB' in the planes ABC and $AB'C$ perpendicular to AC, and let their plane intersect the surface of the sphere in the arc BB'.

To prove that $\angle DAD'$ is measured by the arc BB'.

$\angle DAD'$ is the plane angle of the diedral $BACB'$. (§ 170.)

Whence, $\angle DAD' = \angle BOB'$. (§ 429.)

But $\angle BOB'$ is measured by the arc BB'. (§ 192.)

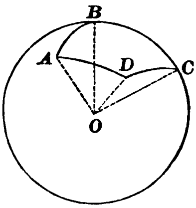
Therefore, $\angle DAD'$ is measured by the arc BB'.

604. Cor. *The angle between two arcs of great circles is the plane angle of the diedral formed by their planes.*

SPHERICAL POLYGONS AND SPHERICAL PYRAMIDS.

Definitions.

605. A *spherical polygon* is a portion of the surface of a sphere bounded by three or more arcs of *great circles*; as $ABCD$.



The bounding arcs are called the *sides* of the spherical polygon.

The *angles* of the spherical polygon are the spherical angles (§ 602) formed by the adjacent sides; and their vertices are called the *vertices* of the spherical polygon.

A *diagonal* is an arc of a great circle joining any two vertices which are not consecutive.

606. The planes of the sides of a spherical polygon form a polyedral, $O\text{–}ABCD$, whose vertex is the centre of the sphere, and whose face angles AOB, BOC, etc., are measured by the sides AB, BC, etc., of the spherical polygon (§ 192).

A spherical polygon is called *convex* when its corresponding polyedral is convex (§ 456).

607. Since the sum of the face angles of any convex polyedral is less than four right angles (§ 463), the sum of their measures is less than a circumference.

That is, *the sum of the sides of a convex spherical polygon is less than the circumference of a great circle.*

608. A *spherical pyramid* is the solid bounded by a spherical polygon and the planes of its sides; as $O\text{–}ABCD$ (§ 605).

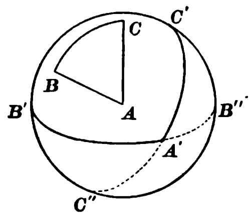
The centre of the sphere is called the *vertex* of the spherical pyramid, and the spherical polygon is called its *base*.

609. The sides of a spherical polygon, being arcs, are usually measured in *degrees*.

610. A *spherical triangle* is a spherical polygon of three sides.

It is called *isosceles, equilateral,* or *right-angled* in the same cases as a plane triangle.

611. If, with the vertices of a spherical triangle as poles, arcs of great circles be described, a spherical triangle is formed which is called the *polar triangle* of the first.



Thus, if A, B, and C are the poles of the arcs $B'C'$, $C'A'$, and $A'B'$, then $A'B'C'$ is the polar triangle of ABC.

612. The circumferences of which $B'C'$, $C'A'$, and $A'B'$ are arcs, form by their intersection *eight* spherical triangles.

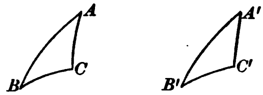
Four of these, i.e., $A'B'C'$, $A'B'C''$, $A'B''C'$, and $A'B''C''$, lie on the hemisphere represented in the figure, and the others lie on the opposite hemisphere.

Of these eight spherical triangles, that is the polar triangle in which the vertex A' homologous to A, lies on the same side of BC as the vertex A; and similarly for the remaining vertices.

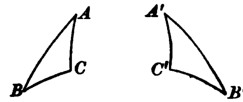
613. Two spherical polygons are *equal* when they can be applied one to the other so as to coincide throughout.

614. Two spherical polygons are equal when the sides and angles of one are equal respectively to the homologous sides and angles of the other, if the equal parts are arranged *in the same order*.

Thus, the spherical triangles ABC and $A'B'C'$ are equal if the sides AB, BC, and CA are equal to $A'B'$, $B'C'$, and $C'A'$, respectively, and the angles A, B, and C to the angles A', B', and C'; for they can evidently be applied one to the other so as to coincide throughout.



615. Two spherical polygons are said to be ***symmetrical*** when the sides and angles of one are equal respectively to the homologous sides and angles of the other, if the equal parts are arranged *in the reverse order*.

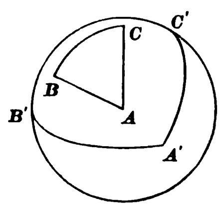


Thus, the spherical triangles ABC and $A'B'C'$ are symmetrical if the sides AB, BC, and CA are equal to $A'B'$, $B'C'$, and $C'A'$, respectively, and the angles A, B, and C to the angles A', B', and C'.

616. It is evident that in general two symmetrical spherical triangles cannot be applied one to the other so as to coincide throughout. (Compare § 636.)

Proposition XII. Theorem.

617. *If one spherical triangle is the polar triangle of another, then the second spherical triangle is the polar triangle of the first.*



Let $A'B'C'$ be the polar triangle of ABC.

To prove that ABC is the polar triangle of $A'B'C'$.

Now B is the pole of the arc $A'C'$. (§ 611.)

Whence, A' lies at a quadrant's distance from B. (§ 592.)

Again, C is the pole of the arc $A'B'$.

Whence, A' lies at a quadrant's distance from C.

Therefore, A' is the pole of the arc BC. (§ 594.)

In like manner, it may be proved that B' is the pole of the arc CA, and C' of the arc AB.

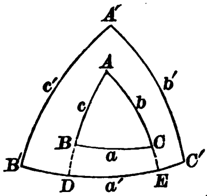
Then, ABC is the polar triangle of $A'B'C'$.

For the homologous vertices A and A' lie on the same side of $B'C'$ (§ 612), and similarly for the remaining vertices.

618. Def. Two spherical triangles, each of which is the polar triangle of the other, are called *polar triangles.*

Proposition XIII. Theorem.

619. *In two polar triangles, each angle of one is measured by the supplement of the side lying opposite the homologous angle of the other.*



Let a, b, c, and a', b', c' denote the sides, expressed in degrees, and A, B, C, and A', B', C' the angles, also expressed in degrees, of the polar triangles ABC and $A'B'C'$.

To prove

$$A = 180° - a', \quad B = 180° - b', \quad C = 180° - c',$$
$$A' = 180° - a, \quad B' = 180° - b, \quad C' = 180° - c.$$

Produce the arcs AB and AC to meet the arc $B'C'$ at D and E.

Then since B' is the pole of the arc AE, and C' of the arc AD, the arcs $B'E$ and $C'D$ are quadrants. (§ 592.)

Therefore, arc $B'E$ + arc $C'D = 180°$.

That is, arc DE + arc $B'C' = 180°$.

But A is the pole of the arc $B'C'$.

Whence, the angle A is measured by the arc DE. (§ 603.)

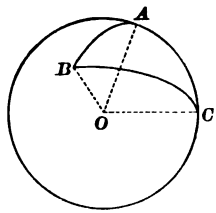
Therefore, $A + a' = 180°$.

Whence, $A = 180° - a'$.

In like manner, the theorem may be proved for any angle of either triangle.

PROPOSITION XIV. THEOREM.

620. *The sum of any two sides of a spherical triangle is greater than the third side.*



Let ABC be a spherical triangle on the surface of a sphere whose centre is O.

To prove $$AB + AC > BC.$$

Draw OA, OB, and OC; then in the triedral $O\text{–}ABC$,

$$\angle AOB + \angle AOC > \angle BOC. \qquad (\S\ 462.)$$

But the sides AB, AC, and BC are the measures of the angles AOB, AOC, and BOC, respectively. (§ 192.)

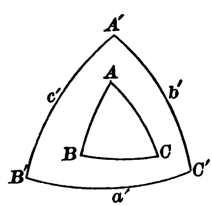
Therefore, $$AB + AC > BC.$$

In like manner, we may prove

$$AB + BC > AC, \text{ and } AC + BC > AB.$$

PROPOSITION XV. THEOREM.

621. *The sum of the angles of a spherical triangle is greater than two, and less than six, right angles.*



Let ABC be a spherical triangle.

To prove $A + B + C > 180°$, and $< 540°$.

Let $A'B'C'$ be the polar triangle of ABC, and denote its sides by a', b', and c'.

Then, $$A = 180° - a',$$
$$B = 180° - b',$$
and $$C = 180° - c'. \quad (\S\ 619.)$$

Adding these equations, we have

$$A + B + C = 540° - (a' + b' + c'). \quad (1)$$

Whence, $A + B + C < 540°$.

Again, the sum of the sides of the spherical triangle $A'B'C'$ is less than the circumference of a great circle. (§ 607.)

That is, $a' + b' + c' < 360°$.

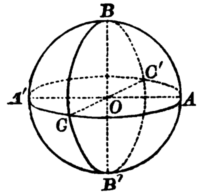
Whence by (1), $A + B + C > 180°$.

622. Cor. I. *A spherical triangle may have one, two, or three right angles, or one, two, or three obtuse angles.*

623. Def. A spherical triangle having two right angles is called a *bi-rectangular triangle.*

A spherical triangle having three right angles is called a *tri-rectangular triangle.*

624. Cor. II. *If three planes be passed through the centre of a sphere in such a way that each is perpendicular to the other two, the surface is divided into eight equal tri-rectangular triangles.*



For each angle of either spherical triangle is a right angle.

Also, each side of either triangle is a quadrant. (§ 191.)

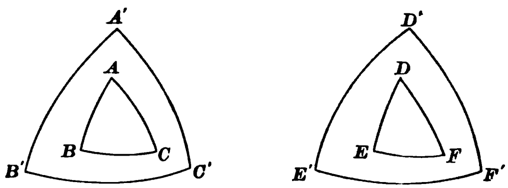
Hence, the spherical triangles are all equal. (§ 614.)

625. Cor. III. *The surface of a sphere is eight times the surface of a tri-rectangular triangle.*

626. DEF. Two spherical polygons on the same sphere, or equal spheres, are said to be *mutually equilateral*, or *mutually equiangular*, when the sides or angles of one are equal respectively to the homologous sides or angles of the other, whether taken in the same or in the reverse order.

PROPOSITION XVI. THEOREM.

627. *If two spherical triangles on the same sphere, or equal spheres, are mutually equiangular, their polar triangles are mutually equilateral.*



Let ABC and DEF be mutually equiangular spherical triangles on the same sphere, or on equal spheres; the angles A and D being homologous.

To prove that their polar triangles, $A'B'C'$ and $D'E'F'$, are mutually equilateral.

The angles A and D are measured by the supplements of the sides $B'C'$ and $E'F'$, respectively. (§ 619.)

But by hypothesis, $\angle A = \angle D$.

Whence, $B'C' = E'F'$. (§ 33, 2.)

In like manner, any two homologous sides of $A'B'C'$ and $D'E'F'$ may be proved equal.

Therefore, $A'B'C'$ and $D'E'F'$ are mutually equilateral.

628. COR. (Converse of Prop. XVI.) *If two spherical triangles on the same sphere, or equal spheres, are mutually equilateral, their polar triangles are mutually equiangular.*

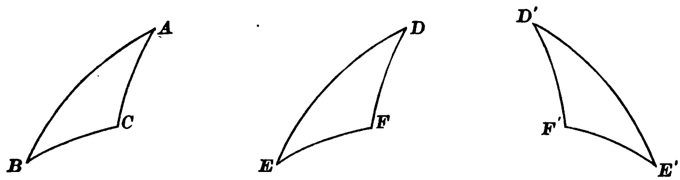
(The proof is left to the student; compare § 627.)

PROPOSITION XVII. THEOREM.

629. *If two spherical triangles on the same sphere, or equal spheres, have two sides and the included angle of one equal respectively to two sides and the included angle of the other,*

I. *They are equal if the equal parts occur in the same order.*

II. *They are symmetrical if the equal parts occur in the reverse order.*



I. Let ABC and DEF be spherical triangles on the same sphere, or equal spheres, having

$$AB = DE,\ AC = DF, \text{ and } \angle A = \angle D.$$

To prove ABC and DEF equal.

Superpose ABC upon DEF so that $\angle A$ shall coincide with $\angle D$; the side AB falling upon DE, and AC upon DF.

Then, since $AB = DE$ and $AC = DF$, B will fall at E, and C at F; and the side BC will coincide with EF. (§ 587.)

Hence, ABC and DEF coincide throughout, and are equal.

II. Let ABC and $D'E'F'$ be spherical triangles on the same sphere, or equal spheres, having

$$AB = D'E',\ AC = D'F', \text{ and } \angle A = \angle D'.$$

To prove ABC and $D'E'F'$ symmetrical.

Construct the spherical triangle DEF symmetrical to $D'E'F'$, having $DE = D'E'$, $DF = D'F'$, and $\angle D = \angle D'$.

Then in the spherical triangles ABC and DEF, we have

$$AB = DE,\ AC = DF, \text{ and } \angle A = \angle D.$$

Whence, ABC and DEF are equal. (§ 629, I.)

Therefore, ABC is symmetrical to $D'E'F'$.

Proposition XVIII. Theorem.

630. *If two spherical triangles on the same sphere, or equal spheres, have a side and two adjacent angles of one equal respectively to a side and two adjacent angles of the other,*

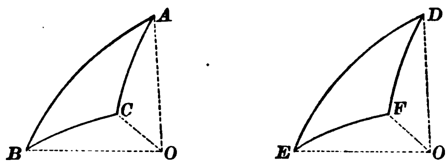
I. *They are equal if the equal parts occur in the same order.*

II. *They are symmetrical if the equal parts occur in the reverse order.*

(The proof is left to the student; compare § 629.)

Proposition XIX. Theorem.

631. *If two spherical triangles on the same sphere, or equal spheres, are mutually equilateral, they are mutually equiangular.*



Let ABC and DEF be mutually equilateral spherical triangles on equal spheres, the sides BC and EF being homologous.

To prove ABC and DEF mutually equiangular.

Let O and O' be the centres of the respective spheres.

Draw OA, OB, OC, $O'D$, $O'E$, and $O'F$.

The triedrals $O\text{–}ABC$ and $O'\text{–}DEF$ have their homologous face angles equal. (§ 192.)

Therefore, diedral $OA =$ diedral $O'D$. (§ 465.)

But the spherical angles BAC and EDF are the plane angles of the diedrals OA and $O'D$. (§ 604.)

Whence, $\angle BAC = \angle EDF$. (§ 432.)

In like manner, any two homologous angles of ABC and DEF may be proved equal.

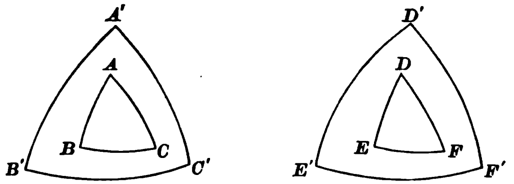
Whence, ABC and DEF are mutually equiangular.

632. Cor. *If two spherical triangles on the same sphere, or equal spheres, are mutually equilateral,*

1. *They are equal if the equal parts occur in the same order.*
2. *They are symmetrical if the equal parts occur in the reverse order.*

Proposition XX. Theorem.

633. *If two spherical triangles on the same sphere, or equal spheres, are mutually equiangular, they are mutually equilateral.*



Let ABC and DEF be mutually equiangular spherical triangles on the same sphere, or equal spheres.

To prove ABC and DEF mutually equilateral.

Let $A'B'C'$ be the polar triangle of ABC, and $D'E'F'$ of DEF.

Then since ABC and DEF are mutually equiangular, $A'B'C'$ and $D'E'F'$ are mutually equilateral. (§ 627.)

Then $A'B'C'$ and $D'E'F'$ are mutually equiangular. (§ 631.)

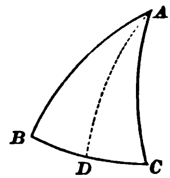
Therefore, their polar triangles, ABC and DEF, are mutually equilateral. (§ 627.)

634. Cor. *If two spherical triangles on the same sphere, or equal spheres, are mutually equiangular,*

1. *They are equal if the equal parts occur in the same order.*
2. *They are symmetrical if the equal parts occur in the reverse order.*

PROPOSITION XXI. THEOREM.

635. *In an isosceles spherical triangle, the angles opposite the equal sides are equal.*



In the spherical triangle ABC, let $AB = AC$.

To prove $\angle B = \angle C$.

Let AD be an arc of a great circle bisecting the side BC.

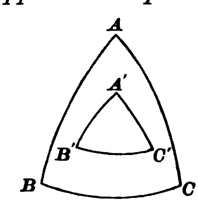
Then in the spherical triangles ABD and ACD, the side AD is common; also, $AB = AC$, and $BD = CD$.

Then ABD and ACD are mutually equiangular. (§ 631.)

Whence, $\angle B = \angle C$.

636. COR. I. *Two symmetrical* (§ 615) *isosceles spherical triangles are equal*; for they can be applied one to the other so as to coincide throughout.

637. COR. II. (Converse of Prop. XXI.) *If two angles of a spherical triangle are equal, the sides opposite are equal.*



In the spherical triangle ABC, let

$$\angle B = \angle C.$$

To prove $AB = AC$.

Let $A'B'C'$ be the polar triangle of ABC.

Then, $A'B'$ is the supplement of $\angle C$, and $A'C'$ of $\angle B$. (§ 619.)

Whence, $A'B' = A'C'$. (§ 33, 2.)

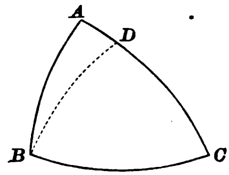
Therefore, $\angle C' = \angle B'$. (§ 635.)

But AB is the supplement of $\angle C'$, and AC of $\angle B'$.

Whence, $AB = AC$.

PROPOSITION XXII. THEOREM.

638. *In any spherical triangle, the greater side lies opposite the greater angle.*



In the spherical triangle ABC, let $\angle ABC$ be $> \angle C$.

To prove $AC > AB$.

Let BD be an arc of a great circle making $\angle CBD = \angle C$.

Then, $BD = CD$. (§ 637.)

But, $AD + BD > AB$. (§ 620.)

Therefore, $AD + CD > AB$.

That is, $AC > AB$.

639. COR. (Converse of Prop. XXII.) *In any spherical triangle, the greater angle lies opposite the greater side.*

In the spherical triangle ABC, let AC be $> AB$.

To prove $\angle ABC > \angle C$.

If $\angle ABC$ were $< \angle C$, AC would be $< AB$. (§ 638.)

And if $\angle ABC$ were equal to $\angle C$, AC would be equal to AB. (§ 637.)

But each of these conclusions is contrary to the hypothesis that AC is $> AB$.

Hence, $\angle ABC > \angle C$.

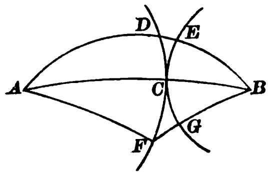
EXERCISES.

1. If the sides of a spherical triangle are 77°, 123°, and 95°, how many degrees are there in each angle of its polar triangle?

2. If the angles of a spherical triangle are 86°, 131°, and 68°, how many degrees are there in each side of its polar triangle?

PROPOSITION XXIII. THEOREM.

640. *The shortest line on the surface of a sphere between two given points is the arc of a great circle, not greater than a semi-circumference, which joins the points.*



Let AB be an arc of a great circle, not greater than a semi-circumference, which joins the given points A and B.

To prove AB the shortest line on the surface of the sphere between A and B.

Let C be any point in the arc AB.

Let DCF and ECG be arcs of small circles, having A and B respectively as poles, and AC and BC as polar distances.

The arcs DCF and ECG have only the point C common.

For let F be any other point in the arc DCF, and draw the arcs of great circles AF and BF.

Then, $\qquad AF + BF > AC + BC. \qquad$ (§ 620.)

Subtracting arc AF from the first member of the inequality, and its equal AC from the second member, we have

$$BF > BC.$$

Therefore, F lies without the small circle ECG, and the arcs DCF and ECG have only the point C common.

We will next prove that the shortest line on the surface of the sphere from A to B must pass through C.

Let $ADEB$ be any line drawn on the surface of the sphere between A and B, not passing through C, and cutting the arcs DCF and ECG at D and E, respectively.

Then, whatever the nature of the line AD, it is evident that an equal line can be drawn from A to C.

In like manner, whatever the nature of the line BE, an equal line can be drawn from B to C.

Hence, a line can be drawn from A to B passing through C, equal to the sum of the lines AD and BE, and consequently less than the line $ADEB$ by the portion DE.

Therefore, no line which does not pass through C can be the shortest line between A and B.

But C is *any* point in the arc AB.

Hence the shortest line from A to B must pass through *every* point of AB.

That is, the arc of a great circle AB is the shortest line which can be drawn on the surface of the sphere between A and B.

EXERCISES.

3. Any point in the arc of a great circle bisecting a spherical angle is equally distant (§ 588) from the sides of the angle.

4. A point on the surface of a sphere, equally distant from the sides of a spherical angle, lies in the arc of a great circle bisecting the angle.

5. The sum of the angles of a spherical hexagon is greater than 8, and less than 12, right angles.

6. The sum of the angles of a spherical polygon of n sides is greater than $2n - 4$, and less than $2n$, right angles.

7. The sides opposite the equal angles of a bi-rectangular triangle are quadrants.

8. State and prove the converse of Ex. 7.

9. Any side of a spherical polygon is less than the sum of the remaining sides.

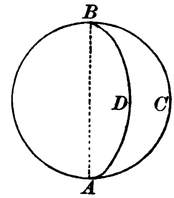
10. The arc of a great circle drawn from the vertex of an isosceles spherical triangle to the middle point of the base, is perpendicular to the base, and bisects the vertical angle.

11. How many degrees are there in the polar distance of a circle, whose plane is $5\sqrt{2}$ units from the centre of the sphere, the diameter of the sphere being 20 units?

12. The polar distance of a circle of a sphere is 60°. If the diameter of the circle is 6, find the diameter of the sphere, and the distance of the circle from its centre.

MEASUREMENT OF SPHERICAL POLYGONS.

641. Def. A *lune* is a portion of the surface of a sphere bounded by two semi-circumferences of great circles; as $ACBD$.



The *angle* of the lune is the angle included between its bounding arcs.

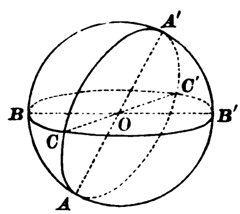
642. It is evident that *two lunes on the same sphere, or equal spheres, are equal when their angles are equal.*

643. A *spherical wedge* is the solid bounded by a lune and the planes of its bounding arcs.

The lune is called the *base* of the spherical wedge.

Proposition XXIV. Theorem.

644. *The spherical triangles corresponding to a pair of vertical triedrals are symmetrical.*



Let AOA', BOB', and COC' be diameters of the sphere AC.

To prove the spherical triangles ABC and $A'B'C'$ symmetrical.

The angles AOB, BOC, and COA are equal respectively to the angles $A'OB'$, $B'OC'$, and $C'OA'$. (§ 39.)

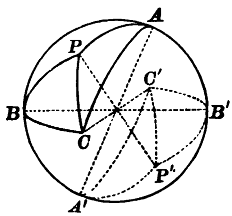
Then, $AB = A'B'$, $BC = B'C'$, and $CA = C'A'$. (§ 192.)

But the equal parts of ABC and $A'B'C'$ are arranged in the reverse order.

Therefore, ABC and $A'B'C'$ are symmetrical. (§ 632, 2.)

PROPOSITION XXV. THEOREM.

645. *Two symmetrical spherical triangles are equivalent.*



Let AA', BB', and CC' be diameters of the sphere AB.

Then the spherical triangles ABC and $A'B'C'$ are symmetrical. (§ 644.)

To prove area ABC = area $A'B'C'$.

Let P be the pole of the small circle passing through the points A, B, and C, and draw the arcs of great circles PA, PB, and PC.

Then, $PA = PB = PC$. (§ 590.)

Draw the diameter of the sphere PP'.

Also, draw the arcs of great circles $P'A'$, $P'B'$, and $P'C'$.

Then the spherical triangles PAB and $P'A'B'$ are symmetrical. (§ 644.)

But the spherical triangle PAB is isosceles.

Therefore, PAB is equal to $P'A'B'$. (§ 636.)

In like manner, we may prove PBC equal to $P'B'C'$, and PCA equal to $P'C'A'$.

Then the sum of the areas of the triangles PAB, PBC, and PCA is equal to the sum of the areas of $P'A'B'$, $P'B'C'$, and $P'C'A'$.

That is, area ABC = area $A'B'C'$.

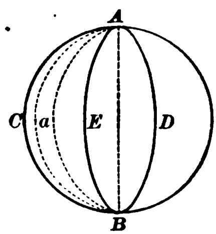
646. SCH. If P and P' fall without the spherical triangles ABC and $A'B'C'$, we should take the sum of the areas of two isosceles spherical triangles, diminished by the area of a third.

PROPOSITION XXVI. THEOREM.

647. *Two lunes on the same sphere, or equal spheres, are to each other as their angles.*

NOTE. The word "*lune*," in the above statement, signifies the *area* of the lune.

CASE I. *When the angles are commensurable.*



Let $ACBD$ and $ACBE$ be two lunes, whose angles CAD and CAE are commensurable.

To prove $$\frac{ACBD}{ACBE} = \frac{\angle CAD}{\angle CAE}.$$

Let CAa be a common measure of the angles CAD and CAE, and let it be contained 5 times in CAD, and 3 times in CAE.

Then, $$\frac{\angle CAD}{\angle CAE} = \frac{5}{3}. \qquad (1)$$

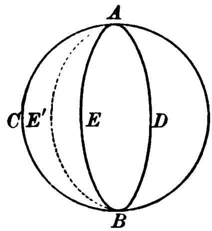
Producing the several arcs of division of the angle CAD to B, the lune $ACBD$ will be divided into 5 parts, and the lune $ACBE$ into 3 parts, all of which parts will be equal. (§ 642.)

Then, $$\frac{ACBD}{ACBE} = \frac{5}{3}. \qquad (2)$$

From (1) and (2), we have,

$$\frac{ACBD}{ACBE} = \frac{\angle CAD}{\angle CAE}.$$

CASE II. *When the angles are incommensurable.*



Let $ACBD$ and $ACBE$ be two lunes, whose angles CAD and CAE are incommensurable.

To prove $$\frac{ACBD}{ACBE} = \frac{\angle CAD}{\angle CAE}.$$

Let $\angle CAD$ be divided into any number of equal parts, and let one of these parts be applied to $\angle CAE$ as a measure.

Since CAD and CAE are incommensurable, a certain number of the parts will extend from AC to AE', leaving a remainder $E'AE$ less than one of the parts.

Produce the arc AE' to B.

Then, $$\frac{ACBD}{ACBE'} = \frac{\angle CAD}{\angle CAE'}. \quad \text{(§ 647, Case I.)}$$

Now let the number of subdivisions of the angle CAD be indefinitely increased.

Then the magnitude of each part will be indefinitely diminished, and the remainder $E'AE$ will approach the limit 0.

Then, $\frac{ACBD}{ACBE'}$ will approach the limit $\frac{ACBD}{ACBE}$,

and $\frac{\angle CAD}{\angle CAE'}$ will approach the limit $\frac{\angle CAD}{\angle CAE}$.

By the Theorem of Limits, these limits are equal. (§ 188.)

Whence, $$\frac{ACBD}{ACBE} = \frac{\angle CAD}{\angle CAE}.$$

648. Cor. I. *The surface of a lune is to the surface of the sphere as the angle of the lune is to four right angles.*

For the surface of a sphere may be regarded as a lune whose angle is equal to four right angles.

649. Cor. II. Let L denote the area of a lune; A the numerical measure of its angle referred to a right angle as the unit; and T the area of a tri-rectangular triangle.

Then the area of the surface of the sphere is $8\,T$. (§ 625.)

Whence, $$\frac{L}{8\,T} = \frac{A}{4}. \qquad \text{(§ 648.)}$$

That is, $$L = 2\,A \times T.$$

Hence, *if the unit of measure for angles is the right angle, the area of a lune is equal to twice its angle, multiplied by the area of a tri-rectangular triangle.*

Thus, if the area of the surface of a sphere is 72, the area of a tri-rectangular triangle is $\frac{1}{8}$ of 72, or 9.

Then, if the angle of a lune on this sphere is 50°, or $\frac{5}{9}$ of a right angle, its area is $\frac{10}{9}$ of 9, or 10.

650. Sch. It may be proved, exactly as in § 647, that

The volume of a spherical wedge is to the volume of the sphere as the angle of the lune which forms its base is to four right angles.

It follows from the above that

If the unit of measure for angles is the right angle, the volume of a spherical wedge is equal to twice the angle of the lune which forms its base, multiplied by the volume of a tri-rectangular pyramid.

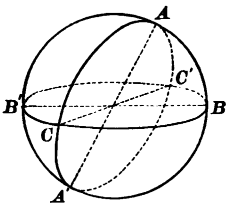
Note. A *tri-rectangular pyramid* is a spherical pyramid whose base is a tri-rectangular triangle.

651. Def. The *spherical excess* of a spherical triangle is the excess of the sum of its angles above two right angles.

Thus, if the angles of a spherical triangle are 65°, 80°, and 95°, its spherical excess is $65° + 80° + 95° - 180°$, or 60°.

Proposition XXVII. Theorem.

652. *If the unit of measure for angles is the right angle, the area of a spherical triangle is equal to its spherical excess, multiplied by the area of a tri-rectangular triangle.*



Let A, B, and C denote the numerical measures of the angles of the spherical triangle ABC referred to a right angle as the unit; and T the area of a tri-rectangular triangle.

To prove area $ABC = (A + B + C - 2) \times T$.

Complete the circumferences $ABA'B'$, $ACA'C'$, and $BCB'C'$; and draw the diameters AA', BB', and CC'.

Then since $ABA'C$ is a lune whose angle is A, we have

$$\text{area } ABC + \text{area } A'BC = 2\,A \times T \text{ (§ 649)}. \qquad (1)$$

And since $BAB'C$ is a lune whose angle is B,

$$\text{area } ABC + \text{area } AB'C = 2\,B \times T. \qquad (2)$$

Now $A'B'C$ and ABC' are symmetrical. (§ 644.)

Whence, area $A'B'C$ = area ABC'. (§ 645.)

Adding area ABC to both members, we have

$$\begin{aligned}\text{area } ABC + \text{area } A'B'C &= \text{area of lune } CBC'A \\ &= 2\,C \times T. \qquad (3)\end{aligned}$$

Adding (1), (2), and (3), and observing that the sum of the areas of ABC, $A'BC$, $AB'C$, and $A'B'C$ is equal to the area of the surface of a hemisphere, or $4\,T$, we have

$$2 \text{ area } ABC + 4\,T = (2\,A + 2\,B + 2\,C) \times T.$$

Then, area $ABC + 2\,T = (A + B + C) \times T$.

Or, area $ABC = (A + B + C - 2) \times T$.

653. Sch. I. Let it be required to find the area of a spherical triangle whose angles are 105°, 80°, and 95°, on a sphere the area of whose surface is 144 sq. in.

The spherical excess of the spherical triangle is 100°, or $\frac{10}{9}$ referred to a right angle as the unit.

And the area of a tri-rectangular triangle is $\frac{1}{8}$ of 144, or 18 sq. in.

Hence, the required area is $\frac{10}{9}$ of 18, or 20 sq. in.

654. Sch. II. It may be proved, as in § 652, that

If the unit of measure for angles is the right angle, the volume of a spherical pyramid is equal to the spherical excess of its base, multiplied by the volume of a tri-rectangular pyramid.

EXERCISES.

13. Find the area of a spherical triangle whose angles are 103°, 112°, and 127°, on a sphere the area of whose surface is 160.

14. Find the volume of a triangular spherical pyramid the angles of whose base are 92°, 119°, and 134°; the volume of the sphere being 192.

15. What is the volume of a spherical wedge the angle of whose base is 127° 30′, if the volume of the sphere is 112 ?

16. The area of a lune is $28\frac{4}{5}$ sq. in. If the area of the surface of the sphere is 120 sq. in., what is the angle of the lune ?

17. What is the ratio of the areas of two spherical triangles on the same sphere, whose angles are 94°, 135°, and 146°, and 87°, 105°, and 118°, respectively ?

18. The area of a spherical triangle two of whose angles are 78° and 99°, is $34\frac{1}{8}$. If the area of the surface of the sphere is 234, what is the other angle ?

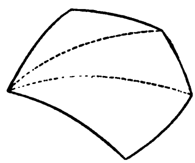
19. The volume of a triangular spherical pyramid the angles of whose base are 105°, 126°, and 147°, is $60\frac{1}{2}$; what is the volume of the sphere ?

20. If two straight lines are tangent to a sphere at the same point, their plane is tangent to the sphere.

21. The sum of the arcs of great circles drawn from any point within a spherical triangle to the extremities of any side, is less than the sum of the other two sides of the triangle.

Proposition XXVIII. Theorem.

655. *If the unit of measure for angles is the right angle, the area of any spherical polygon is equal to the sum of its angles, diminished by as many times two right angles as the figure has sides less two, multiplied by the area of a tri-rectangular triangle.*



Let K denote the area of any spherical polygon; n the number of its sides; s the sum of its angles referred to a right angle as the unit; and T the area of a tri-rectangular triangle.

To prove $\qquad K = [s - 2\,(n - 2)] \times T.$

The spherical polygon may be divided into spherical triangles by drawing diagonals from any vertex; the number of such spherical triangles being equal to the number of sides of the spherical polygon, less two.

Now the area of each spherical triangle is equal to the sum of its angles, less two right angles, multiplied by T. (§ 652.)

Hence, the sum of the areas of the spherical triangles is equal to the sum of their angles, diminished by as many times two right angles as there are triangles, multiplied by T.

But the number of triangles is $n - 2$.

Therefore, the area of the spherical polygon is equal to the sum of its angles, diminished by $n - 2$ times two right angles, multiplied by T.

That is, $\qquad K = [s - 2\,(n - 2)] \times T.$

656. Sch. It may be proved, as in § 655, that

If the unit of measure for angles is the right angle, the volume of any spherical pyramid is equal to the sum of the angles of its base, diminished by as many times two right angles as the base has sides less two, multiplied by the volume of a tri-rectangular pyramid.

657. Cor. Let P denote the volume of a spherical pyramid; and let K denote the area of the base, n the number of its sides, and s the sum of its angles referred to a right angle as the unit.

Let V represent the volume of the sphere; S the area of its surface; T the area of a tri-rectangular triangle; and T' the volume of a tri-rectangular pyramid.

Then, $P = [s - 2(n-2)] \times T'$, (§ 656.)

and $K = [s - 2(n-2)] \times T.$ (§ 655.)

Also, $V = 8\,T'$, and $S = 8\,T.$

Whence by division,

$$\frac{P}{K} = \frac{T'}{T}, \text{ and } \frac{V}{S} = \frac{T'}{T}.$$

Therefore, $$\frac{P}{K} = \frac{V}{S}.$$

That is, *the volume of a spherical pyramid is to its base as the volume of the sphere is to its surface.*

EXERCISES.

22. Find the area of a spherical hexagon whose angles are 120°, 139°, 148°, 155°, 162°, and 167°, on a sphere the area of whose surface is 280.

23. Find the volume of a pentagonal spherical pyramid the angles of whose base are 109°, 128°, 137°, 153°, and 158°; the volume of the sphere being 180.

24. The arcs of great circles bisecting the angles of a spherical triangle meet in a point equally distant from the sides of the triangle. (Exs. 3, 4, p. 337.)

25. A circle may be inscribed in any spherical triangle.

26. State and prove the theorem for spherical triangles analogous to Prop. IX., Book I.

27. State and prove the theorem for spherical triangles analogous to Prop. V., Book I.

28. State and prove the theorem for spherical triangles analogous to Prop. LI., Book I.

29. The volume of a quadrangular spherical pyramid, the angles of whose base are 110°, 122°, 135°, and 146°, is $12\frac{4}{5}$ cu. ft. What is the volume of the sphere?

30. The area of a spherical pentagon, four of whose angles are 112°, 131°, 138°, and 168°, is 27. If the area of the surface of the sphere is 120, what is the other angle?

31. If the side AB of a spherical triangle ABC is equal to a quadrant, and the side BC is less than a quadrant, prove that $\angle A$ is less than 90°.

32. If PA, PB, and PC are three equal arcs of great circles drawn from a point P to the circumference of a great circle ABC, prove that P is the pole of ABC.

33. The spherical polygons corresponding to a pair of vertical polyedrals are symmetrical.

34. Either angle of a spherical triangle is greater than the difference between 180° and the sum of the other two angles.

35. If a polyedron be circumscribed about each of two equal spheres, the volumes of the polyedrons are to each other as the areas of their surfaces.

36. If ABC and $A'B'C'$ are a pair of polar triangles on a sphere whose centre is O, prove that the radius OA' is perpendicular to the plane OBC.

37. The intersection of two spheres is a circle, whose centre lies in the line joining the centres of the spheres, and whose plane is perpendicular to this line.

38. The distance between the centres of two spheres, whose radii are 25 in. and 17 in., respectively, is 28 in. Find the diameter of their circle of intersection, and the distance of its plane from the centre of each sphere.

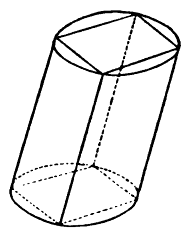
662. Cor. II. Let S denote the lateral area, T the total area, H the altitude, and R the radius of the base, of a cylinder of revolution.

Then, $$S = 2\pi RH. \qquad (§ 368.)$$

And $$T = 2\pi RH + 2\pi R^2 \qquad (§ 371.)$$
$$= 2\pi R(H + R).$$

Proposition II. Theorem.

663. *The volume of a circular cylinder is equal to the product of its base and altitude.*



Let V denote the volume, B the area of the base, and H the altitude, of a circular cylinder.

To prove $$V = B \times H.$$

Inscribe in the cylinder a prism whose base is a regular polygon; let V' denote its volume, and B' the area of its base.

Then since the altitude of the prism is H, we have
$$V' = B' \times H. \qquad (§ 507.)$$

Now let the number of faces of the prism be indefinitely increased.

Then, V' approaches the limit V. (§ 659, 2.)

And $B' \times H$ approaches the limit $B \times H$. (§ 363, II.)

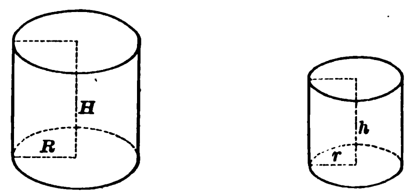
Therefore, $$V = B \times H. \qquad (§ 188.)$$

664. Cor. Let V denote the volume, H the altitude, and R the radius of the base, of a circular cylinder.

Then, $$V = \pi R^2 H. \qquad (§ 371.)$$

PROPOSITION III. THEOREM.

665. *The lateral or total areas of two similar cylinders of revolution are to each other as the squares of their altitudes, or as the squares of the radii of their bases; and their volumes are to each other as the cubes of their altitudes, or as the cubes of the radii of their bases.*



Let S and s denote the lateral areas, T and t the total areas, V and v the volumes, H and h the altitudes, and R and r the radii of the bases, of two similar cylinders of revolution (§ 555).

To prove $$\frac{S}{s} = \frac{T}{t} = \frac{H^2}{h^2} = \frac{R^2}{r^2},$$

and $$\frac{V}{v} = \frac{H^3}{h^3} = \frac{R^3}{r^3}.$$

The generating rectangles are similar.

Whence, $$\frac{H}{h} = \frac{R}{r} \qquad (§ 253, \text{II.})$$

$$= \frac{H+R}{h+r}. \qquad (§ 239.)$$

Then,

$$\frac{S}{s} = \frac{2\,\pi RH}{2\,\pi rh} \quad (§ 662) = \frac{R}{r} \times \frac{R}{r} = \frac{R^2}{r^2} = \frac{H^2}{h^2};$$

$$\frac{T}{t} = \frac{2\,\pi R\,(H+R)}{2\,\pi r\,(h+r)} \quad (§ 662) = \frac{R}{r} \times \frac{R}{r} = \frac{R^2}{r^2} = \frac{H^2}{h^2};$$

and

$$\frac{V}{v} = \frac{\pi R^2 H}{\pi r^2 h} \quad (§ 664) = \frac{R^2}{r^2} \times \frac{R}{r} = \frac{R^3}{r^3} = \frac{H^3}{h^3}.$$

THE CONE.

Definitions.

666. A pyramid is said to be *inscribed in a cone* when its base is inscribed in the base of the cone, and its vertex coincides with the vertex of the cone.

A pyramid is said to be *circumscribed about a cone* when its base is circumscribed about the base of the cone, and its vertex coincides with the vertex of the cone.

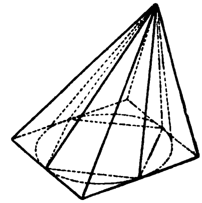
A frustum of a pyramid is said to be *inscribed in a frustum of a cone* when its bases are inscribed in the bases of the frustum of the cone.

A frustum of a pyramid is said to be *circumscribed about a frustum of a cone* when its bases are circumscribed about the bases of the frustum of the cone.

The *lateral area* of a cone, or frustum of a cone, is the area of its lateral surface.

The *slant height* of a cone of revolution is the straight line drawn from the vertex to any point in the circumference of the base.

The *slant height* of a frustum of a cone of revolution is that portion of the slant height of the cone included between the bases of the frustum.



667. It may be proved, exactly as in § 659, that

If a pyramid whose base is a regular polygon be inscribed in, or circumscribed about, a circular cone (§ 565), *and the number of its faces be indefinitely increased,*

1. *The lateral area of the pyramid approaches the lateral area of the cone as a limit.*

2. *The volume of the pyramid approaches the volume of the cone as a limit.*

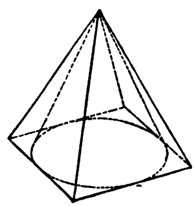
668. It follows from § 667 that

If a frustum of a pyramid whose base is a regular polygon be inscribed in, or circumscribed about, a frustum of a circular cone, and the number of its faces be indefinitely increased,

1. *The lateral area of the frustum of the pyramid approaches the lateral area of the frustum of the cone as a limit.*
2. *The volume of the frustum of the pyramid approaches the volume of the frustum of the cone as a limit.*

PROPOSITION IV. THEOREM.

669. *The lateral area of a cone of revolution is equal to the circumference of its base, multiplied by one-half its slant height.*



Let S denote the lateral area, C the circumference of the base, and L the slant height, of a cone of revolution.

To prove $S = C \times \frac{1}{2} L.$

Circumscribe about the cone a regular pyramid; let S denote its lateral area, and C' the perimeter of its base.

Then since the sides of the base of the pyramid are bisected at their points of contact, the slant height of the pyramid is the same as the slant height of the cone. (§ 515.)

Therefore, $S' = C' \times \frac{1}{2} L.$ (§ 523.)

Now let the number of faces of the pyramid be indefinitely increased.

Then, S' approaches the limit S. (§ 667, 1.)

And $C' \times \frac{1}{2} L$ approaches the limit $C \times \frac{1}{2} L$. (§ 363, I.)

Therefore, $S = C \times \frac{1}{2} L.$ (§ 188.)

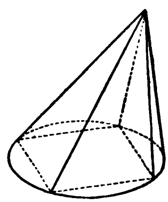
670. Cor. Let S denote the lateral area, T the total area, L the slant height, and R the radius of the base, of a cone of revolution.

Then, $S = 2\pi R \times \frac{1}{2} L$ (§ 368) $= \pi RL$.

And $T = \pi RL + \pi R^2$ (§ 371) $= \pi R(L + R)$.

Proposition V. Theorem.

671. *The volume of a circular cone is equal to one-third the product of its base and altitude.*



Let V denote the volume, B the area of the base, and H the altitude, of a circular cone.

To prove $V = \frac{1}{3} B \times H$.

Inscribe in the cone a pyramid whose base is a regular polygon; let V' denote its volume, and B' the area of its base.

Then, $V' = \frac{1}{3} B' \times H$. (§ 527.)

Now let the number of faces of the pyramid be indefinitely increased.

Then, V' approaches the limit V. (§ 667, 2.)

And $\frac{1}{3} B' \times H$ approaches the limit $\frac{1}{3} B \times H$. (§ 363, II.)

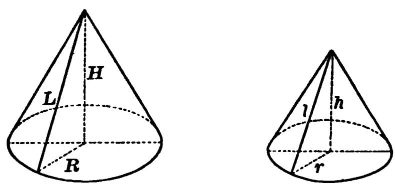
Therefore, $V = \frac{1}{3} B \times H$. (§ 188.)

672. Cor. Let V denote the volume, H the altitude, and R the radius of the base, of a circular cone.

Then, $V = \frac{1}{3} \pi R^2 H$. (§ 371.)

PROPOSITION VI. THEOREM.

673. *The lateral or total areas of two similar cones of revolution are to each other as the squares of their slant heights, or as the squares of their altitudes, or as the squares of the radii of their bases; and their volumes are to each other as the cubes of their slant heights, or as the cubes of their altitudes, or as the cubes of the radii of their bases.*



Let S and s denote the lateral areas, T and t the total areas, V and v the volumes, L and l the slant heights, H and h the altitudes, and R and r the radii of the bases, of two similar cones of revolution (§ 568).

To prove $$\frac{S}{s} = \frac{T}{t} = \frac{L^2}{l^2} = \frac{H^2}{h^2} = \frac{R^2}{r^2},$$

and $$\frac{V}{v} = \frac{L^3}{l^3} = \frac{H^3}{h^3} = \frac{R^3}{r^3}.$$

The generating triangles are similar.

Whence, $$\frac{L}{l} = \frac{R}{r} = \frac{H}{h} = \frac{L+R}{l+r}.$$

Then,

$$\frac{S}{s} = \frac{\pi RL}{\pi rl} \quad (\S\ 670) = \frac{R}{r} \times \frac{R}{r} = \frac{R^2}{r^2} = \frac{L^2}{l^2} = \frac{H^2}{h^2};$$

$$\frac{T}{t} = \frac{\pi R(L+R)}{\pi r(l+r)} \quad (\S\ 670) = \frac{R}{r} \times \frac{R}{r} = \frac{R^2}{r^2} = \frac{L^2}{l^2} = \frac{H^2}{h^2};$$

and

$$\frac{V}{v} = \frac{\frac{1}{3}\pi R^2 H}{\frac{1}{3}\pi r^2 h} \quad (\S\ 672) = \frac{R^2}{r^2} \times \frac{R}{r} = \frac{R^3}{r^3} = \frac{L^3}{l^3} = \frac{H^3}{h^3}.$$

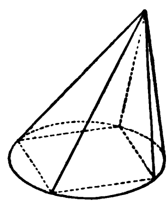
670. Cor. Let S denote the lateral area, T the total area, L the slant height, and R the radius of the base, of a cone of revolution.

Then, $S = 2\pi R \times \frac{1}{2} L$ (§ 368) $= \pi RL$.

And $T = \pi RL + \pi R^2$ (§ 371) $= \pi R(L + R)$.

PROPOSITION V. THEOREM.

671. *The volume of a circular cone is equal to one-third the product of its base and altitude.*



Let V denote the volume, B the area of the base, and H the altitude, of a circular cone.

To prove $V = \frac{1}{3} B \times H$.

Inscribe in the cone a pyramid whose base is a regular polygon; let V' denote its volume, and B' the area of its base.

Then, $V' = \frac{1}{3} B' \times H$. (§ 527.)

Now let the number of faces of the pyramid be indefinitely increased.

Then, V' approaches the limit V. (§ 667, 2.)

And $\frac{1}{3} B' \times H$ approaches the limit $\frac{1}{3} B \times H$. (§ 363, II.)

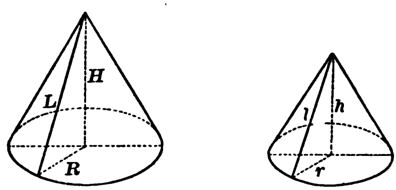
Therefore, $V = \frac{1}{3} B \times H$. (§ 188.)

672. Cor. Let V denote the volume, H the altitude, and R the radius of the base, of a circular cone.

Then, $V = \frac{1}{3} \pi R^2 H$. (§ 371.)

PROPOSITION VI. THEOREM.

673. *The lateral or total areas of two similar cones of revolution are to each other as the squares of their slant heights, or as the squares of their altitudes, or as the squares of the radii of their bases; and their volumes are to each other as the cubes of their slant heights, or as the cubes of their altitudes, or as the cubes of the radii of their bases.*



Let S and s denote the lateral areas, T and t the total areas, V and v the volumes, L and l the slant heights, H and h the altitudes, and R and r the radii of the bases, of two similar cones of revolution (§ 568).

To prove $$\frac{S}{s}=\frac{T}{t}=\frac{L^2}{l^2}=\frac{H^2}{h^2}=\frac{R^2}{r^2},$$

and $$\frac{V}{v}=\frac{L^3}{l^3}=\frac{H^3}{h^3}=\frac{R^3}{r^3}.$$

The generating triangles are similar.

Whence, $$\frac{L}{l}=\frac{R}{r}=\frac{H}{h}=\frac{L+R}{l+r}.$$

Then,

$$\frac{S}{s}=\frac{\pi RL}{\pi rl} \quad (\S\ 670)=\frac{R}{r}\times\frac{R}{r}=\frac{R^2}{r^2}=\frac{L^2}{l^2}=\frac{H^2}{h^2};$$

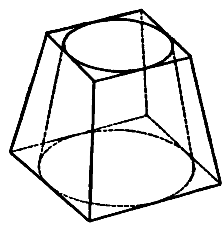
$$\frac{T}{t}=\frac{\pi R\,(L+R)}{\pi r\,(l+r)} \quad (\S\ 670)=\frac{R}{r}\times\frac{R}{r}=\frac{R^2}{r^2}=\frac{L^2}{l^2}=\frac{H^2}{h^2};$$

and

$$\frac{V}{v}=\frac{\frac{1}{3}\,\pi R^2H}{\frac{1}{3}\,\pi r^2h} \quad (\S\ 672)=\frac{R^2}{r^2}\times\frac{R}{r}=\frac{R^3}{r^3}=\frac{L^3}{l^3}=\frac{H^3}{h^3}.$$

Proposition VII. Theorem.

674. *The lateral area of a frustum of a cone of revolution is equal to one-half the sum of the circumferences of its bases, multiplied by its slant height.*



Let S denote the lateral area, C and c the circumferences of the bases, and L the slant height, of a frustum of a cone of revolution.

To prove $S = \frac{1}{2}(C + c) \times L.$

Circumscribe about the frustum a frustum of a regular pyramid.

Let S' denote its lateral area, and C' and c' the perimeters of its bases.

Then since the sides of the bases of the frustum of the pyramid are bisected at their points of contact, the slant height of the frustum of the pyramid is the same as the slant height of the frustum of the cone. (§ 515.)

Therefore, $S' = \frac{1}{2}(C' + c') \times L.$ (§ 524.)

Now let the number of faces of the frustum of the pyramid be indefinitely increased.

Then, S' approaches the limit S. (§ 668, 1.)

And

$\frac{1}{2}(C' + c') \times L$ approaches the limit $\frac{1}{2}(C + c) \times L.$ (§ 363, I.)

Therefore, $S = \frac{1}{2}(C + c) \times L.$ (§ 188.)

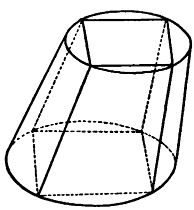
675. Cor. I. *The lateral area of a frustum of a cone of revolution is equal to the circumference of a section equally distant from its bases, multiplied by its slant height.*

676. Cor. II. Let S denote the lateral area, L the slant height, and R and r the radii of the bases, of a frustum of a cone of revolution.

Then, $S = \frac{1}{2}(2\pi R + 2\pi r) \times L$ (§ 368) $= \pi(R + r)L$.

Proposition VIII. Theorem.

677. *The volume of a frustum of a circular cone is equal to the sum of its bases and a mean proportional between its bases, multiplied by one-third its altitude.*



Let V denote the volume, B and b the areas of the bases, and H the altitude, of a frustum of a circular cone.

To prove $\quad V = (B + b + \sqrt{B \times b}) \times \frac{1}{3}H.$

Inscribe in the frustum a frustum of a pyramid whose base is a regular polygon; let V' denote its volume, and B' and b' the areas of its bases.

Then, $\quad V' = (B' + b' + \sqrt{B' \times b'}) \times \frac{1}{3}H.$ (§ 532.)

Now let the number of faces of the frustum of the pyramid be indefinitely increased.

Then, $\quad V'$ approaches the limit V. (§ 668, 2.)

And $(B' + b' + \sqrt{B' \times b'}) \times \frac{1}{3}H$ approaches the limit

$(B + b + \sqrt{B \times b}) \times \frac{1}{3}H.$ (§ 363, II.)

Whence, $\quad V = (B + b + \sqrt{B \times b}) \times \frac{1}{3}H.$ (§ 188.)

678. Cor. Let V denote the volume, H the altitude, and R and r the radii of the bases, of a frustum of a circular cone.

Then, $B = \pi R^2$, and $b = \pi r^2$. (§ 371.)

Whence, $\sqrt{B \times b} = \sqrt{\pi^2 R^2 r^2} = \pi R r$.

Therefore,
$$V = (\pi R^2 + \pi r^2 + \pi R r) \times \tfrac{1}{3} H$$
$$= \tfrac{1}{3}\pi (R^2 + r^2 + Rr) H.$$

EXERCISES.

1. Find the lateral area, total area, and volume of a cylinder of revolution, the diameter of whose base is 18, and whose altitude is 16.

2. Find the lateral area, total area, and volume of a cone of revolution, the radius of whose base is 7, and whose slant height is 25.

3. Find the lateral area, total area, and volume of a frustum of a cone of revolution, the diameters of whose bases are 16 and 6, and whose altitude is 12.

4. Find the altitude and diameter of the base of a cylinder of revolution, whose lateral area is 168π and volume 504π.

5. Find the volume of a cone of revolution, whose slant height is 29 and lateral area 580π.

6. Find the lateral area of a cone of revolution, whose volume is 320π and altitude 15.

7. Find the volume of a cylinder of revolution, whose total area is 170π and altitude 12.

8. The altitude of a cone of revolution is 27, and the radius of its base is 16. What is the diameter of the base of an equivalent cylinder of revolution, whose altitude is 16?

9. The area of the entire surface of a frustum of a cone of revolution is 306π, and the radii of its bases are 11 and 5. Find its lateral area and volume.

10. The volume of a frustum of a cone of revolution is 6020π, its altitude is 60, and the radius of its lower base is 15. Find the radius of its upper base and its lateral area.

11. Find the altitude and lateral area of a cone of revolution, whose volume is 800π, and whose slant height is to the diameter of its base as 13 to 10.

THE SPHERE.

DEFINITIONS.

679. A *zone* is a portion of the surface of a sphere included between two parallel planes.

The circumferences of the circles which bound the zone are called the *bases*, and the perpendicular distance between their planes the *altitude*.

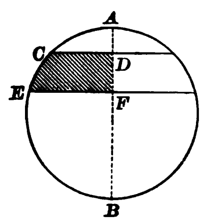
A *zone of one base* is a zone one of whose bounding planes is tangent to the sphere.

680. A *spherical segment* is a portion of a sphere included between two parallel planes.

The circles which bound it are called the *bases*, and the perpendicular distance between them the *altitude*.

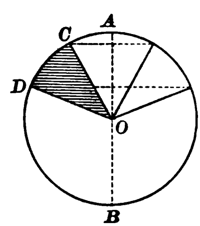
A *spherical segment of one base* is a spherical segment one of whose bounding planes is tangent to the sphere.

681. If the semicircle $ACEB$ be revolved about its diameter AB as an axis, and CD and EF are perpendicular to AB, the arc CE generates a zone whose altitude is DF, and the figure $CEFD$ a spherical segment whose altitude is DF.



The arc AC generates a zone of one base, and the figure ACD a spherical segment of one base.

682. If a semicircle be revolved about its diameter as an axis, any sector generates a solid called a *spherical sector*.

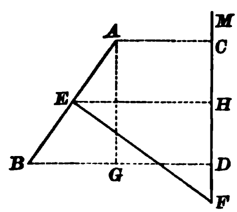


Thus, if the semicircle $ACDB$ be revolved about AB as an axis, the sector OCD generates a spherical sector.

The zone generated by the arc CD is called the *base* of the spherical sector.

PROPOSITION IX. THEOREM.

683. *The area generated by the revolution of a straight line about an axis in its plane, not parallel to itself, is equal to its projection on the axis, multiplied by the circumference of a circle, whose radius is the perpendicular erected at the middle point of the line and terminating in the axis.*



Let the straight line AB be revolved about the axis FM in its plane, not parallel to itself.

Let CD be the projection of AB on FM, and EF the perpendicular erected at the middle point of AB, terminating in the axis.

To prove area $AB^* = CD \times 2\pi EF$. (§ 368.)

Draw AG perpendicular to BD, and EH perpendicular to CD.

The surface generated by AB is the lateral surface of a frustum of a cone of revolution, whose bases are generated by AC and BD.

Then, area $AB = AB \times 2\pi EH$. (§ 675.)

But the triangles ABG and EFH are similar. (§ 261.)

Whence, $$\frac{AB}{AG} = \frac{EF}{EH}.$$

Therefore, $AB \times EH = AG \times EF$ (§ 231.)

$= CD \times EF.$

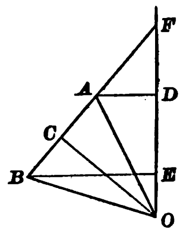
Then, area $AB = CD \times 2\pi EF$.

---

* The expression "area AB" is used to denote the *area generated by AB.*

PROPOSITION X. THEOREM.

684. *If an isosceles triangle be revolved about an axis in its plane, not parallel to its base, which passes through its vertex without intersecting its surface, the volume generated is equal to the area generated by the base, multiplied by one-third the altitude.*



Let the isosceles triangle OAB be revolved about the axis OF in its plane, not parallel to AB; and draw the altitude OC.

To prove vol. OAB* = area $AB \times \frac{1}{3} OC$.

Draw AD and BE perpendicular to OF; and produce BA to meet OF at F.

Now, vol. OBF = vol. OBE + vol. BEF

$$= \tfrac{1}{3}\pi\overline{BE}^2 \times OE + \tfrac{1}{3}\pi\overline{BE}^2 \times EF \qquad (\S\ 672.)$$

$$= \tfrac{1}{3}\pi\overline{BE}^2 \times (OE + EF) = \tfrac{1}{3}\pi\overline{BE}^2 \times OF.$$

But $BE \times OF = OC \times BF$; for each expresses twice the area of the triangle OBF. (§ 313.)

Hence, vol. $OBF = \frac{1}{3}\pi BE \times OC \times BF$.

But $\pi BE \times BF$ is the area generated by BF. (§ 670.)

Whence, vol. OBF = area $BF \times \frac{1}{3} OC$. (1)

Also, vol. OAF = area $AF \times \frac{1}{3} OC$. (2)

Subtracting (2) from (1), we have

$$\text{vol. } OAB = (\text{area } BF - \text{area } AF) \times \tfrac{1}{3} OC$$
$$= \text{area } AB \times \tfrac{1}{3} OC.$$

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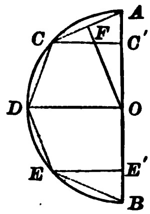
* The expression "vol. OAB" is used to denote the volume generated by OAB.

Proposition XI. Theorem.

685. *The area of the surface of a sphere is equal to its diameter multiplied by the circumference of a great circle.*



Let O be the centre of the semicircle ADB; and let the sphere be generated by the revolution of ADB about AB as an axis.

Let R denote the radius of the sphere.

To prove that the area of the surface of the sphere is

$$AB \times 2\,\pi R.$$

Divide the arc ADB into four equal arcs, AC, CD, DE, and EB; and draw the chords AC, CD, DE, and EB.

Also, draw CC', DO, and EE' perpendicular to AB, and OF perpendicular to AC.

Then, area $AC = AC' \times 2\,\pi OF$. (§ 683.)

Also, area $CD = C'O \times 2\,\pi OF$; etc.

Adding these equations, we have

area generated by broken line $ACDEB$

$$= (AC' + C'O + \text{etc.}) \times 2\,\pi OF = AB \times 2\,\pi OF.$$

Now let the subdivisions of the arc ADB be bisected indefinitely.

Then the area generated by the broken line $ACDEB$ approaches the area generated by the arc ADB as a limit. (§ 363, I.)

And $AB \times 2\,\pi OF$ approaches $AB \times 2\,\pi R$ as a limit. (§ 364, 1.)

Then area generated by arc $ADB = AB \times 2\,\pi R$. (§ 188.)

686. Cor. I. Let S denote the area of the surface of a sphere, R its radius, and D its diameter.

Then, $$S = 2R \times 2\pi R = 4\pi R^2.$$

That is, *the area of the surface of a sphere is equal to the square of its radius multiplied by* 4π.

Again, $$S = \pi \times (2R)^2 = \pi D^2.$$

That is, *the area of the surface of a sphere is equal to the square of its diameter multiplied by* π.

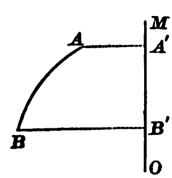
687. Cor. II. Let S and S' denote the areas of the surfaces of two spheres, R and R' their radii, and D and D' their diameters.

Then, $$\frac{S}{S'} = \frac{4\pi R^2}{4\pi R'^2} = \frac{R^2}{R'^2},$$

and $$\frac{S}{S'} = \frac{\pi D^2}{\pi D'^2} = \frac{D^2}{D'^2}. \qquad (\S\ 686.)$$

That is, *the areas of the surfaces of two spheres are to each other as the squares of their radii, or as the squares of their diameters.*

688. Cor. III. Let O be the centre of the arc AB; and draw AA' and BB' perpendicular to the diameter OM.



It may be proved, as in § 685, that the area generated by the revolution of the arc AB about OM as an axis is equal to

$$A'B' \times 2\pi R,$$

where R is the radius of the arc.

That is, *the area of a zone is equal to its altitude multiplied by the circumference of a great circle.*

EXERCISES.

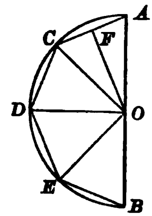
12. Find the area of the surface of a sphere whose radius is 12.

13. Find the area of a zone whose altitude is 13, if the radius of the sphere is 16.

14. The surface of a sphere is equivalent to four great circles.

Proposition XII. Theorem.

689. ***The volume of a sphere is equal to the area of its surface multiplied by one-third its radius.***



Let O be the centre of the semicircle ADB; and let the sphere be generated by the revolution of ADB about AB as an axis.

Let R denote the radius of the sphere.

To prove that the volume of the sphere is equal to the area of its surface, multiplied by $\frac{1}{3} R$.

Divide the arc ADB into four equal arcs, AC, CD, DE, and EB; and draw the chords AC, CD, DE, and EB.

Draw OC, OD, and OE; also, OF perpendicular to AC.

Then, vol. $OAC =$ area $AC \times \frac{1}{3} OF$. (§ 684.)

Also, vol. $OCD =$ area $CD \times \frac{1}{3} OF$; etc.

Adding these equations, we have

volume generated by polygon $ACDEB$

$$= (\text{area } AC + \text{area } CD + \text{etc.}) \times \tfrac{1}{3} OF$$
$$= \text{area } ACDEB \times \tfrac{1}{3} OF.$$

Now let the subdivisions of the arc ADB be bisected indefinitely.

Then the volume generated by the polygon $ACDEB$ approaches the volume generated by the semicircle ADB as a limit. (§ 363, II.)

And the area generated by $ACDEB \times \frac{1}{3} OF$ approaches the area generated by the arc $ADB \times \frac{1}{3} R$ as a limit.

(§§ 363, I., 364, 1.)

Then volume generated by ADB

$= \text{area generated by arc } ADB \times \frac{1}{3} R.$ (§ 188.)

690. Cor. I. Let V denote the volume of a sphere, R its radius, and D its diameter.

Then, $V = 4\,\pi R^2 \times \frac{1}{3} R$ (§ 686) $= \frac{4}{3}\,\pi R^3$.

That is, *the volume of a sphere is equal to the cube of its radius multiplied by* $\frac{4}{3}\,\pi$.

Again, $V = \frac{1}{6}\,\pi \times (2\,R)^3 = \frac{1}{6}\,\pi D^3$.

That is, *the volume of a sphere is equal to the cube of its diameter multiplied by* $\frac{1}{6}\,\pi$.

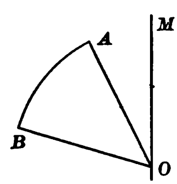
691. Cor. II. Let V and V' denote the volumes of two spheres, R and R' their radii, and D and D' their diameters.

Then,
$$\frac{V}{V'} = \frac{\frac{4}{3}\,\pi R^3}{\frac{4}{3}\,\pi R'^3} = \frac{R^3}{R'^3},$$
and
$$\frac{V}{V'} = \frac{\frac{1}{6}\,\pi D^3}{\frac{1}{6}\,\pi D'^3} = \frac{D^3}{D'^3}. \qquad (§ 690.)$$

That is, *the volumes of two spheres are to each other as the cubes of their radii, or as the cubes of their diameters.*

692. Cor. III. Let OAB be a sector of a circle.

It may be proved, as in § 689, that the volume generated by the revolution of OAB about the diameter OM as an axis, is equal to the area generated by the arc AB, multiplied by $\frac{1}{3} R$, where R is the radius of the arc.



That is, *the volume of a spherical sector is equal to the area of the zone which forms its base, multiplied by one-third the radius of the sphere.*

693. Cor. IV. Let h denote the altitude of the zone which forms the base of the spherical sector of § 692.

Then, $\text{vol. } OAB = h \times 2\,\pi R \times \frac{1}{3} R$ (§ 688.)

$= \frac{2}{3}\,\pi R^2 h.$

694. Cor. V. Let P denote the volume of a spherical pyramid, and K the area of its base; also, let V denote the volume, S the area of the surface, and R the radius, of the sphere.

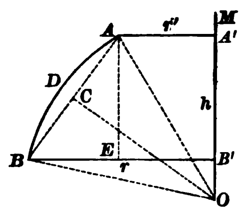
Then, $\frac{P}{K} = \frac{V}{S}$ (§ 657) $= \frac{\frac{4}{3}\pi R^3}{4\pi R^2}$ (§§ 686, 690) $= \frac{1}{3} R$.

Whence, $P = K \times \frac{1}{3} R$.

That is, *the volume of a spherical pyramid is equal to the area of its base multiplied by one-third the radius of the sphere.*

Proposition XIII. Problem.

695. *To find the volume of a spherical segment.*



Let O be the centre of the arc ADB.

Draw AA' and BB' perpendicular to the diameter OM.

To find the volume of the spherical segment generated by the revolution of the figure $ADBB'A'$ about OM as an axis.

Let $AA' = r'$, $BB' = r$, $A'B' = h$, and $OA = R$.

Draw OA, OB, and AB; also, draw OC perpendicular to AB, and AE perpendicular to BB'.

Now, vol. $ADBB'A'$ = vol. $ACBD$ + vol. $ABB'A'$. (1)

Also, vol. $ACBD$ = vol. $OADB$ − vol. OAB.

But, vol. $OADB = \frac{2}{3}\pi R^2 h$. (§ 693.)

And vol. OAB = area $AB \times \frac{1}{3} OC$ (§ 684.)

$= h \times 2\pi OC \times \frac{1}{3} OC$ (§ 683.)

$= \frac{2}{3}\pi \overline{OC}^2 h.$

Then, $\quad \text{vol. } ACDB = \frac{2}{3}\pi R^2 h - \frac{2}{3}\pi \overline{OC}^2 h$

$$= \frac{2}{3}\pi (R^2 - \overline{OC}^2)\, h.$$

But, $\quad R^2 - \overline{OC}^2 = \overline{AC}^2 \qquad$ (§ 274.)

$$= (\tfrac{1}{2} AB)^2 = \tfrac{1}{4}\overline{AB}^2.$$

Then, $\quad \text{vol. } ACDB = \frac{2}{3}\pi \times \frac{1}{4}\overline{AB}^2 \times h = \frac{1}{6}\pi \overline{AB}^2 h.$

Now, $\quad \overline{AB}^2 = \overline{BE}^2 + \overline{AE}^2 \qquad$ (§ 273.)

$$= (r - r')^2 + h^2.$$

Then, $\quad \text{vol. } ACDB = \frac{1}{6}\pi [(r - r')^2 + h^2]\, h.$

Also, $\quad \text{vol. } ABB'A' = \frac{1}{3}\pi (r^2 + r'^2 + rr')\, h. \qquad$ (§ 678.)

Substituting in (1), we have

$$\begin{aligned}
&\text{vol. } ADBB'A' \\
&= \tfrac{1}{6}\pi [(r - r')^2 + h^2]\, h + \tfrac{1}{3}\pi (r^2 + r'^2 + rr')\, h \\
&= \tfrac{1}{6}\pi (r^2 - 2\, rr' + r'^2 + h^2 + 2\, r^2 + 2\, r'^2 + 2\, rr')\, h \\
&= \tfrac{1}{6}\pi (3\, r^2 + 3\, r'^2 + h^2)\, h \\
&= \tfrac{1}{2}\pi (r^2 + r'^2)\, h + \tfrac{1}{6}\pi h^3.
\end{aligned}$$

696. Cor. If r denotes the radius of the base, and h the altitude, of a spherical segment of one base, its volume is

$$\tfrac{1}{2}\pi r^2 h + \tfrac{1}{6}\pi h^3.$$

EXERCISES.

15. Find the volume of a sphere whose radius is 12.

16. Find the volume of a spherical sector the altitude of whose base is 12, the diameter of the sphere being 25.

17. Find the volume of a spherical segment, the radii of whose bases are 4 and 5, and whose altitude is 9.

18. Find the radius and volume of a sphere the area of whose surface is 324π.

19. Find the diameter and area of the surface of a sphere whose volume is $\frac{1125}{2}\pi$.

20. The surface of a sphere is equivalent to the lateral surface of its circumscribed cylinder.

21. The volume of a sphere is two-thirds the volume of its circumscribed cylinder.

22. A spherical cannon-ball 9 in. in diameter is dropped into a cubical box filled with water, whose depth is 9 in. How many cubic inches of water will be left in the box?

23. What is the angle of the base of a spherical wedge whose volume is $\frac{40}{3}\pi$, if the radius of the sphere is 4?

24. Find the area of a spherical triangle whose angles are 125°, 133°, and 156°, on a sphere whose radius is 10.

25. Find the volume of a quadrangular spherical pyramid, the angles of whose base are 107°, 118°, 134°, and 146°; the diameter of the sphere being 12.

26. The surface of a sphere is equivalent to two-thirds the entire surface of its circumscribed cylinder.

27. The volume of a cylinder of revolution is equal to its lateral area multiplied by one-half the radius of its base.

28. Prove Prop. IX. when the straight line is parallel to the axis.

29. Find the area of the surface and the volume of a sphere, inscribed in a cube the area of whose surface is 486.

30. How many spherical bullets, each $\frac{5}{8}$ in. in diameter, can be formed from five pieces of lead, each in the form of a cone of revolution, the radius of whose base is 5 in., and whose altitude is 8 in.?

31. Find the volume of a sphere circumscribing a cube whose volume is 64.

32. A cylindrical vessel, 8 in. in diameter, is filled to the brim with water. A ball is immersed in it, displacing water to the depth of $2\frac{1}{4}$ in. Find the diameter of the ball.

33. The radii of the bases of two similar cylinders of revolution are 24 and 44, respectively. If the lateral area of the first cylinder is 720, what is the lateral area of the second?

34. The slant heights of two similar cones of revolution are 9 and 15, respectively. If the volume of the second cone is 625, what is the volume of the first?

35. The total areas of two similar cylinders of revolution are 32 and 162, respectively. If the volume of the second cylinder is 1458, what is the volume of the first?

36. The volumes of two similar cones of revolution are 343 and 512, respectively. If the lateral area of the first cone is 196, what is the lateral area of the second?

37. If a sphere 6 in. in diameter weighs 351 ounces, what is the weight of a sphere of the same material whose diameter is 10 in.?

38. If a sphere whose radius is $12\frac{1}{2}$ in. weighs 3125 lb., what is the radius of a sphere of the same material whose weight is $819\frac{1}{5}$ lb.?

39. The altitude of a frustum of a cone of revolution is $3\frac{1}{2}$, and the radii of its bases are 5 and 3. What is the diameter of an equivalent sphere?

40. Find the radius of a sphere whose surface is equivalent to the entire surface of a cylinder of revolution, whose altitude is $10\frac{1}{2}$ and radius of base 3.

41. A cubical piece of lead, the area of whose entire surface is 384 sq. in., is melted and formed into a cone of revolution, the radius of whose base is 12 in. Find the altitude of the cone.

42. The volume of a cylinder of revolution is equal to the area of its generating rectangle, multiplied by the circumference of a circle whose radius is the distance to the axis from the centre of the rectangle.

43. The volume of a cone of revolution is equal to its lateral area, multiplied by one-third the distance of any element of its lateral surface from the centre of the base.

44. Two zones on the same sphere, or on equal spheres, are to each other as their altitudes.

45. The area of a zone of one base is equal to the area of the circle whose radius is the chord of its generating arc.

46. If the radius of a sphere is R, what is the area of a zone of one base, whose generating arc is 45°?

47. If the altitude of a cone of revolution is 24, and its slant height 25, find the total area of an inscribed cylinder, the radius of whose base is 2.

48. Find the area of the surface and the volume of a sphere circumscribing a cylinder of revolution, the radius of whose base is 9, and whose altitude is 24.

49. A cone of revolution is circumscribed about a sphere whose diameter is two-thirds the altitude of the cone. Prove that its lateral surface and volume are, respectively, three-halves and nine-fourths the surface and volume of the sphere.

50. If the altitude of a cone of revolution is three-fourths the radius of its base, its volume is equal to its lateral area multiplied by one-fifth the radius of its base.

51. A cone of revolution is inscribed in a sphere whose diameter is $\frac{4}{3}$ the altitude of the cone. Prove that its lateral surface and volume are, respectively, $\frac{3}{8}$ and $\frac{9}{32}$ the surface and volume of the sphere.

52. If the radius of a sphere is 25, find the lateral area and volume of an inscribed cone, the radius of whose base is 24.

53. If the volume of a sphere is $\frac{500}{3}\pi$, find the lateral area and volume of a circumscribed cone whose altitude is 18.

54. Find the volume of a spherical segment of one base whose altitude is 6, the diameter of the sphere being 30.

55. A circular sector whose central angle is 45° and radius 12, revolves about a diameter perpendicular to one of its bounding radii. Find the volume of the spherical sector generated.

56. Two equal small circles of a sphere are equally distant from the centre.

57. A square whose area is A, revolves about its diagonal as an axis. Find the area of the entire surface, and the volume, of the solid generated.

58. How many cubic feet of metal are there in a hollow cylindrical tube 18 ft. long, whose outer diameter is 8 in., and thickness 1 in.?

59. The altitude of a cone of revolution is 9 in. At what distances from the vertex must it be cut by planes parallel to its base, in order that it may be divided into three equivalent parts?

60. Given the radius of the base, R, and the total area, T, of a cylinder of revolution, to find its volume.

61. Given the diameter of the base, D, and the volume, V, of a cylinder of revolution, to find its lateral area and total area.

62. Given the altitude, H, and the volume, V, of a cone of revolution, to find its lateral area.

63. Given the slant-height, L, and the lateral area, S, of a cone of revolution, to find its volume.

64. Given the area of the surface of a sphere, S, to find its volume.

65. Given the volume of a sphere, V, to find the area of its surface.

66. A right triangle whose legs are a and b revolves about its hypotenuse as an axis. Find the area of the entire surface, and the volume, of the solid generated.

67. The parallel sides of a trapezoid are 12 and 33, respectively, and its non-parallel sides are 10 and 17. Find the volume generated by the revolution of the trapezoid about its longest side as an axis.

68. Find the diameter of a sphere in which the area of the surface and the volume are expressed by the same numbers.

69. An equilateral triangle, whose side is a, revolves about one of its sides as an axis. Find the area of the entire surface, and the volume, of the solid generated.

70. An equilateral triangle, whose altitude is h, revolves about one of its altitudes as an axis. Find the area of the surface, and the volume, of the solids generated by the triangle, and by its inscribed circle.

71. Find the lateral area and volume of a cylinder of revolution, whose altitude is equal to the diameter of its base, inscribed in a cone of revolution whose altitude is h, and radius of base r.

72. Find the lateral area and volume of a cylinder of revolution, whose altitude is equal to the diameter of its base, inscribed in a sphere whose radius is r.

73. An equilateral triangle, whose side is a, revolves about a straight line drawn through one of its vertices parallel to the opposite side. Find the area of the entire surface, and the volume, of the solid generated.

74. If the radius of a sphere is R, find the circumference and area of a small circle, whose distance from the centre is h.

75. The outer diameter of a spherical shell is 9 in., and its thickness is 1 in. What is its weight, if a cubic inch of the metal weighs $\frac{1}{4}$ lb.?

76. A regular hexagon, whose side is a, revolves about its longest diagonal as an axis. Find the area of the entire surface, and the volume, of the solid generated.

77. The sides AB and BC of a rectangle $ABCD$ are 5 and 8, respectively. Find the volumes generated by the revolution of the triangle ACD about the sides AB and BC as axes.

78. The sides of a triangle are 17, 25, and 28. Find the volume generated by the revolution of the triangle about its longest side as an axis.

79. The cross-section of a tunnel $2\frac{1}{2}$ miles in length is in the form of a rectangle 6 yd. wide and 4 yd. high, surmounted by a semicircle whose diameter is equal to the width of the rectangle. How many cubic yards of material were taken out in its construction?

80. A frustum of a circular cone is equivalent to three cones, whose common altitude is the altitude of the frustum, and whose bases are the lower base, the upper base, and a mean proportional between the bases of the frustum. (§ 677.)

81. The volume of a cone of revolution is equal to the area of its generating triangle, multiplied by the circumference of a circle whose radius is the distance to the axis from the intersection of the medians of the triangle.

82. If the earth be regarded as a sphere whose radius is R, what is the area of the zone visible from a point whose height above the surface is H?

83. The sides AB and BC of an acute-angled triangle ABC, are $\sqrt{241}$ and 10, respectively. Find the volume generated by the revolution of the triangle about an axis in its plane, not intersecting its surface, whose distances from A, B, and C are 2, 17, and 11, respectively.

84. A projectile consists of two hemispheres, connected by a cylinder of revolution. If the altitude and diameter of the base of the cylinder are 8 in. and 7 in., respectively, find the number of cubic inches in the projectile.

85. A tapering hollow iron column, 1 in. thick, is 24 ft. long, 10 in. in outside diameter at one end, and 8 in. in diameter at the other. How many cubic inches of metal were used in its construction?

86. If any triangle be revolved about an axis in its plane, not parallel to its base, which passes through its vertex without intersecting its surface, the volume generated is equal to the area generated by the base, multiplied by one-third the altitude.

87. If any triangle be revolved about an axis which passes through its vertex parallel to its base, the volume generated is equal to the area generated by the base, multiplied by one-third the altitude.

88. A segment of a circle whose bounding arc is a quadrant, and whose radius is r, revolves about a diameter parallel to its bounding chord. Find the area of the entire surface, and the volume, of the solid generated.

89. Find the area of the surface of the sphere circumscribing a regular tetraedron whose edge is 8.

ANSWERS

TO THE

NUMERICAL EXERCISES.

NOTE. Those answers are omitted which, if given, would destroy the utility of the problem.

Page 16.

6. 24°. **7.** 63° 30′, 26° 30′. **8.** 22° 30′, 157° 30′.

9. 37°.

Page 41.

27. $A = 20°$, $B = 60°$, $C = 100°$.

Page 44.

31. $A = 112° 30′$, $B = 33° 45′$, $C = 33° 45′$.

Page 68.

96. 7.

Page 97.

15. 52° 30′. **17.** 96°. **18.** 164°.

Page 98.

21. 28° 45′. **22.** 44° 30′. **23.** 12°.

Page 99.

25. 54° 30′. **26.** 178°.

Page 101.

29. 112° 30′. **42.** 83°, 89° 30′, 97°, 90° 30′, 74° 30′.

Page 102.

55. $\angle AED = 14° 30′$, $\angle AFB = 10° 30′$.

58. 114° 30′, 89° 30′, 65° 30′, 90° 30′.

Page 103.

70. $97° 30'$, $89° 30'$, $82° 30'$, $90° 30'$.

Page 126.

1. 112. **2.** 42. **3.** $\frac{24}{27}$. **4.** 63.

Page 132.

5. $3\frac{1}{5}$, $2\frac{4}{5}$. **6.** $11\frac{2}{3}$, $18\frac{2}{3}$.

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7. $19\frac{3}{5}$, $25\frac{1}{5}$.

Page 141.

9. 4 ft. 6 in. **10.** 12.

Page 145.

12. 15. **13.** 37 ft. 1 in. **14.** 47 ft. 6 in. **15.** 4.33 +.
16. 1 ft. 9.06 + in.

Page 147.

18. 58. **19.** 24.

Page 153.

21. 21. **24.** 26. **27.** 48. **28.** 10. **29.** $13\frac{1}{3}$.
30. 12.726 +. **31.** 45.

Page 154.

33. $17\frac{2}{3}$. **36.** 50. **40.** $\sqrt{129}$, $2\sqrt{21}$, $\sqrt{201}$. **41.** $3\frac{1}{3}$.

Page 155.

46. 36. **48.** 63. **49.** 3 and 4; $1\frac{4}{5}$ and $3\frac{1}{5}$. **55.** 24.
56. 17. **57.** 21, 28. **58.** $8\sqrt{3}$. **59.** 12, 4.

Page 156.

61. $3\sqrt{3}$. **62.** 14. **69.** 21. **72.** 70 and 99; 65 and 117.

Page 165.

1. 4 : 3.

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2. $30\frac{5}{8}$ ft. **3.** 8 ft. 9 in. **4.** 14, 12. **5.** 6 ft. 11 in., 20 ft. 9 in.
6. 6 ft. 7 in.

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7. 6 sq. ft. 60 sq. in.

Page 172.

8. 2 sq. ft. 48 sq. in. **9.** 243.

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11. 210; $16\frac{4}{5}$, $24\frac{2}{17}$, 15. **12.** 73. **13.** 117.

Page 177.

18. $\frac{25}{4}\sqrt{3}$. **19.** $3\sqrt{3}$. **20.** 34. **22.** 120. **25.** 210. **26.** 18.

Page 178.

27. $1\frac{1}{2}$ ft. **28.** 6. **29.** $4\sqrt{3}$. **30.** 1260. **34.** 120. **35.** 17. **36.** 624. **38.** 540. **39.** 28. **42.** $4\frac{1}{3}$. **43.** 30, 16.

Page 179.

44. $36\frac{1}{4}$. **47.** $\frac{15}{2}\sqrt{2}$, $11\sqrt{2}$. **48.** 54. **51.** 39, 45. **52.** 1010. **53.** 336.

Page 207.

29. 9. **30.** 88 : 121. **31.** 13. **32.** $\frac{9}{2}\sqrt{2}$.

Page 211.

34. 15.708, 19.635. **35.** Area, 452.3904. **36.** Circumference, 50.2656. **49.** 35.6048. **50.** 35.81424. **51.** 9.827.

Page 212.

52. 10.2102. **53.** 72. **54.** 150.7968. **55.** 1.2732. **56.** 201.0624. **57.** 18.8496. **58.** 50.2656. **59.** 37.6992, 9.4248. **60.** 25.1328, 35.5377. **61.** 9.06. **62.** 1306.9056 sq. ft. **63.** 120.99 ft. **64.** 57 in. **65.** $57.295°+$. **66.** 2.658. **67.** 5.64.

Page 224.

55. $10\sqrt{7}$.

Page 225.

61. 6, 8. **65.** $1\frac{3}{16}$.

Page 228.

91. 480.

Page 275.

1. 4:3. 2. 2:5.

Page 277.

4. 42. 5. 1 ft. 9 in. 6. $34\frac{21}{64}$ cu. in.; $63\frac{3}{8}$ sq. in. 7. 574.
8. 1008. 9. 12 and 7. 10. 1944. 13. 17.

Page 279.

14. Volume, $50\sqrt{3}$.

Page 280.

16. Volume, $\frac{243}{2}\sqrt{3}$.

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19. $\sqrt{273}$, $18\sqrt{237}$, $180\sqrt{3}$. 20. $\frac{1}{2}\sqrt{118}$, $3\sqrt{109}$, 15.
21. $\sqrt{97}$, $12\sqrt{93}$, $72\sqrt{3}$. 22. $4\sqrt{39}$, $504\sqrt{3}$, $936\sqrt{3}$.
23. $6\sqrt{3}$, $56\sqrt{26}$, $503\frac{1}{3}$. 24. $4\sqrt{10}$, $72\sqrt{39}$, $672\sqrt{3}$.
25. 150. 26. 320. 27. 2400 sq. in. 28. $3\frac{193}{432}$ cu. ft.
29. 770. 30. Volume, $48\sqrt{5}$. 31. 840. 32. 36 sq. in.
33. 12 in.

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34. 512, 384. 35. 1705. 36. 144. 37. 10, 1.
38. 700, 1568. 39. $\frac{7}{3}\sqrt{57}$, $640\sqrt{3}$. 40. $42\sqrt{91}$, $624\sqrt{3}$.
41. 240, $\frac{190}{3}\sqrt{119}$. 42. 108, $21\sqrt{39}$. 43. 768, 2340.

Page 295.

49. 50. 52. $4\sqrt{3}$, $\frac{2}{3}\sqrt{2}$. 53. 15. 59. 582.

Page 296.

65. 9600 lb. 66. $168\sqrt{3}$, $15\sqrt{219}$. 67. 5700 cu. yd.
68. $\frac{15}{4}\sqrt{35}$.

Page 308.

73. 3456 cu. in. 74. 6 ft. 75. 4 ft. 6 in. 76. $5\sqrt[3]{4}$ in.
77. 960, 3072. 78. 128. 79. 12. 80. 6. 84. $36\sqrt{3}$.

Page 337.

11. 45°. 12. $4\sqrt{3}$, $\sqrt{3}$.

Page 344.

13. 36. 14. 44. 15. $39\frac{2}{3}$. 16. 86° 24′. 17. 3:2. 18. 108°.
19. 220.

Page 346.

22. $66\frac{1}{2}$. 23. $36\frac{1}{4}$.

Page 347.

29. 60 cu. ft. 30. 153°. 38. 30 in., 8 in., 20 in.

Page 358.

1. 288π, 450π, 1296π. 2. 175π, 224π, 392π. 3. 143π, 216π, 388π
4. 14, 12. 5. 2800π.
6. 136π. 7. 300π. 8. 24. 9. 160π, 536π.
10. 4, 1159π. 11. 24, 260π.

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12. 576π. 13. 416π.

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15. 2304π. 16. 1250π. 17. 306π. 18. Volume, 972π.
19. Area of surface, 225π. 22. 347.2956 cu. in.

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23. 56° 15′. 24. 130π. 25. 58π. 29. 81π, $\frac{243}{2}\pi$.
30. 8192. 31. $32\pi\sqrt{3}$. 32. 6 in. 33. 2420.
34. 135. 35. 128. 36. 256. 37. 1625 oz. 38. 8 in.

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39. 7. 40. $4\frac{1}{2}$. 41. $\dfrac{32}{3\pi}$ in. 46. $\pi R^2(2-\sqrt{2})$.
47. $\frac{536}{7}\pi$. 48. 900π, 4500π.

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52. 960π, 6144π. 53. $\frac{585}{4}\pi$, $\frac{675}{2}\pi$. 54. 468π.
55. $576\pi\sqrt{2}$. 57. $\pi A\sqrt{2}$, $\frac{1}{6}\pi A\sqrt{2A}$. 58. 2.7489 +.
59. $3\sqrt[3]{9}$ in., $3\sqrt[3]{18}$ in. 60. $\dfrac{RT-2\pi R^3}{2}$.

61. $\dfrac{4V}{D}$, $\dfrac{8V+\pi D^3}{2D}$. 62. $\dfrac{\sqrt{9V^2+3\pi H^3V}}{H}$.

63. $\dfrac{S^2\sqrt{\pi^2L^4-S^2}}{3\pi^2L^3}$. 64. $\dfrac{S\sqrt{S}}{6\sqrt{\pi}}$. 65. $\sqrt[3]{36\pi V^2}$.

66. $\dfrac{\pi(a+b)ab}{\sqrt{a^2+b^2}}$, $\dfrac{\pi a^2b^2}{3\sqrt{a^2+b^2}}$. 67. 1216π.

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69. $\pi a^2 \sqrt{3}$, $\frac{1}{4} \pi a^3$.

70. By triangle, πh^2, $\frac{1}{9} \pi h^3$; by inscribed circle, $\frac{4}{9} \pi h^2$, $\frac{4}{81} \pi h^3$.

71. $\dfrac{4 \pi r^2 h^2}{(2r+h)^2}$, $\dfrac{2 \pi r^3 h^3}{(2r+h)^3}$.

72. $2 \pi r^2$, $\frac{1}{2} \pi r^3 \sqrt{2}$. 73. $2 \pi a^2 \sqrt{3}$, $\frac{1}{2} \pi a^3$. 75. 67.3698 + lb.

76. $2 \pi a^2 \sqrt{3}$, πa^3. 77. $\frac{640}{3} \pi$, $\frac{490}{3} \pi$. 78. 2100π.

79. 167803.68.

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82. $\dfrac{2 \pi R^2 H}{R+H}$. 83. 1440π. 84. 487.4716.

85. 7238.2464. 88. $2 \pi r^2 (1+\sqrt{2})$, $\frac{1}{3} \pi r^3 \sqrt{2}$.

89. 96π.